UNIFORMITY OF THE UNCOVERED SET OF RANDOM WALK AND CUTOFF FOR LAMPLIGHTER CHAINS

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We show that the measure on markings of $\mathbf{Z}_n^d$, $d \geq 3$, with elements of $\{0,1\}$ given by iid fair coin flips on the range $\mathcal{R}$ of a random walk X run until time T and 0 otherwise becomes indistinguishable from the uniform measure on such markings at the threshold $T = \frac{1}{2}T_{\text{cov}}(\mathbf{Z}_n^d)$. As a consequence of our methods, we show that the total variation mixing time of the random walk on the lamplighter graph $\mathbf{Z}_2 \wr \mathbf{Z}_n^d$, $d \geq 3$, has a cutoff with threshold $\frac{1}{2}T_{\text{cov}}(\mathbf{Z}_n^d)$. We give a general criterion under which both of these results hold; other examples for which this applies include bounded degree expander families, the intersection of an infinite supercritical percolation cluster with an increasing family of balls, the hypercube, and the Caley graph of the symmetric group generated by transpositions. The proof also yields precise asymptotics for the decay of correlation in the uncovered set.

1. Introduction. Suppose $G = (V, E)$ is a finite, connected graph and X is a lazy random walk on G. This means that X is the Markov chain with state space V and transition kernel

$$p(x, y; G) = \mathbf{P}_x[X(1) = y] = \begin{cases} \frac{1}{2} \text{ if } x = y, \\ \frac{1}{2\deg(x)} \text{ if } \{x, y\} \in E. \end{cases}$$

Let

$$\tau_{\text{cov}}(G) = \min\{t \geq 1 : V \text{ is contained in the range of } X|[0,t]\}$$

be the *cover time* and let $T_{\text{cov}}(G) = \mathbf{E}_\pi \tau_{\text{cov}}(G)$ be the *expected cover time*. Here and hereafter, a subscript of π indicates that X is started from stationarity. Let $\tau(y) = \min\{t \geq 1 : X(t) = y\}$ be the first time X hits y and

$$T_{\text{hit}}(G) = \max_{x,y \in V} \mathbf{E}_x \tau(y)$$

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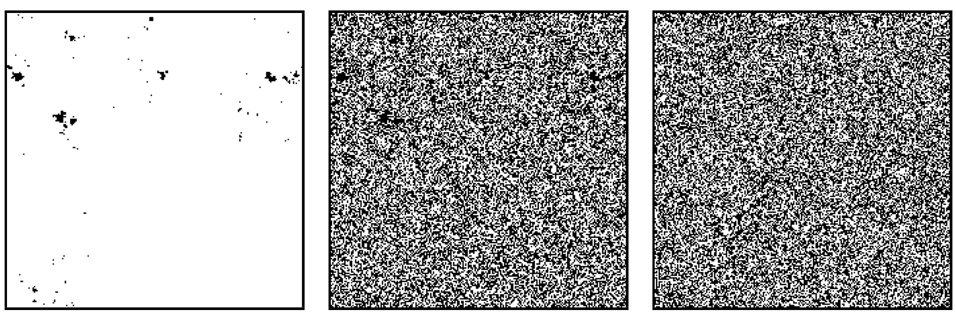
be the *maximal hitting time*. If (G_n) is a sequence of graphs with $T_{\text{hit}}(G_n) = o(T_{\text{cov}}(G_n))$ then a result of Aldous (Theorem 2, [2]) implies that $T_{\text{cov}}(G_n)$ has a threshold around its mean: $T_{\text{cov}}(G_n)/\mathbf{E}\tau_{\text{cov}}(G_n) = 1 + o(1)$. Many sequences of graphs satisfy this condition, for example $\mathbf{Z}_n^d$ for $d \geq 2$, $\mathbf{Z}_2^n$, and the complete graph K_n. When Aldous' condition holds, the set

$$\mathcal{L}(\alpha; G_n) = \{x \in V_n : \tau(x) \geq \alpha T_{\text{cov}}(G_n)\}$$

of *α-late points*, i.e. points hit after time $\alpha T_{\text{cov}}(G_n)$, $\alpha \in (0,1)$, often has an interesting structure. The case $G_n = \mathbf{Z}_n^2$ was first studied by Brummelhuis and Hilhorst in [8] where it is shown that $\mathbf{E}|\mathcal{L}(\alpha; \mathbf{Z}_n^2)|$ has growth exponent $2(1-\alpha)$ and that points in $\mathcal{L}_n(\alpha; \mathbf{Z}_n^2)$ are positively correlated. This suggests that $\mathcal{L}(\alpha; G_n)$ has a fractal structure and exhibits clustering. These statements were made precise by Dembo, Peres, Rosen, and Zeitouni in [13] where they show that the growth exponent of $|\mathcal{L}(\alpha; \mathbf{Z}_n^2)|$ is $2(1-\alpha)$ *with high probability* in addition to making a rigorous quantification of the clustering phenomenon.



(a) $\mathcal{L}(\frac{1}{2}, \mathbf{Z}_n^2)$ (b) $\mathcal{L}(\frac{1}{2}, \mathbf{Z}_n^2)$ marked with iid coin flips (c) $\mathbf{Z}_n^2$ marked with iid coin flips

FIG 1. *The subset $\mathcal{L}(\frac{1}{2}, \mathbf{Z}_n^2)$ of $\mathbf{Z}_n^2$ consisting of those points unvisited by a random walk X run for $\frac{1}{2}T_{\text{cov}}(\mathbf{Z}_n^2)$, where $T_{\text{cov}}(\mathbf{Z}_n^2)$ is the expected number of steps required for X to cover $\mathbf{Z}_n^2$, exhibits clustering. Consequently, the marking of $\mathbf{Z}_n^2$ by elements of $\{0,1\}$ given by the results of iid coin flips on the range of X at time $\frac{1}{2}T_{\text{cov}}(\mathbf{Z}_n^2)$ and zero otherwise can be distinguished from a uniform marking.*

If G_n is either K_n or $\mathbf{Z}_n^d$ for $d \geq 3$ then it is also true that $\log|\mathcal{L}(\alpha; G_n)| \sim (1-\alpha)\log|V_n|$ with high probability. In contrast to the case of $\mathbf{Z}_n^2$, $\mathcal{L}(\alpha; K_n)$ does not exhibit clustering and is "uniformly random" in the sense that conditional on $s_0 = |\mathcal{L}(\alpha; K_n)|$, all subsets of K_n of size s_0 are equally likely. The rapid decay of correlation in $\mathcal{L}(\alpha; \mathbf{Z}_n^d)$ for $d \geq 3$ determined by Brummelhuis and Hilhorst [8] indicates that the clustering phenomenon is

also not present in this case and leads one to speculate that $\mathcal{L}(\alpha;\mathbf{Z}_n^d)$ is likewise in some sense "uniformly random."

The purpose of this article is to quantify the degree to which this holds. We use as our measure of uniformity the following statistical test. Let $\mathcal{R}(\alpha;G)$ be the (random) subset of V covered by X at time $\alpha T_{\mathrm{cov}}(G)$ and let $\mu(\cdot;\alpha,G)$ be the probability measure on $\mathcal{X}(G)=\{f\colon V\to\{0,1\}\}$ given by first sampling $\mathcal{R}(\alpha;G)$ then setting

$$f(x)=\begin{cases}\xi(x) \text{ if } x\in\mathcal{R}(\alpha;G),\\ 0 \text{ otherwise,}\end{cases}$$

where $(\xi(x):x\in V)$ is a collection of iid variables such that $\mathbf{P}[\xi(x)=0]=\mathbf{P}[\xi(x)=1]=\frac{1}{2}$. The question we are interested in is:

How large does $\alpha\in(0,1)$ need to be so that $\mu(\cdot;\alpha,G)$ is indistinguishable from the uniform measure $\nu(\cdot;G)$ on $\mathcal{X}(G)$?

It must be that $\alpha\geq 1/2$ in the case of $\mathbf{Z}_n^d$ for $d\geq 2$ since if $\alpha<1/2$ then

$$\frac{|\mathcal{L}(\alpha;\mathbf{Z}_n^d)|-\frac{1}{2}n^d}{n^{d/2}}\to\infty \text{ as } n\to\infty.$$

In particular, the deviations of the number of zeros from $n^d/2$ for such α far exceeds that in the uniform case. By Theorem 2 of [2], it is also true that $\alpha\leq 1$ since if $\alpha>1$ then with high probability $|\mathcal{L}(\alpha;\mathbf{Z}_n^d)|=0$. The main result of this article is that the threshold for indistinguishability for any sequence of graphs (G_n) with uniformly bounded maximal degree and $\lim_{n\to\infty}|V_n|=\infty$ is $\alpha=\frac{1}{2}$ provided random walk on (G_n) is *uniformly locally transient* and either satisfies a Harnack inequality or whose Green's function exhibits sufficiently fast decay.

We need the following definitions in order to give a precise statement of our results. The *total variation mixing time* of G is

$$T_{\mathrm{mix}}(\epsilon;G)=\min\{t\geq 0:\max_{x\in V}\|p^t(x,\cdot;G)-\pi\|_{TV}\leq\epsilon\}$$

where $p^t(x,y;G)=\mathbf{P}_x[X(t)=y]$ is the t-step transition kernel of X started at x,

$$\|\mu-\nu\|_{TV}=\max_{A\subseteq V}|\mu(A)-\nu(A)|=\frac{1}{2}\sum_{x\in V}|\mu(x)-\nu(x)|$$

is the *total variation distance* between the measures μ, ν on V, and π is the stationary distribution of X. The *uniform mixing time* of G is

$$T_{\text{mix}}^U(\epsilon; G) = \min \left\{ t \geq 0 : \max_{x,y \in V} \left| \frac{p^t(x,y;G)}{\pi(y)} - 1 \right| \leq \epsilon \right\}.$$

It is a basic fact ([4], [21], see also Proposition 3.3) that $T_{\text{mix}}^U(\epsilon; G)$ is within a factor of $\log |V|$ of $T_{\text{mix}}(\epsilon; G)$, however, for many graphs this factor is constant. Whenever we omit ϵ and write $T_{\text{mix}}(G), T_{\text{mix}}^U(G)$ it is understood that $\epsilon = \frac{1}{4}$. The *Green's function* of G is

$$g(x,y;G) = \sum_{t=1}^{T_{\text{mix}}^U(G)} p^t(x,y;G),$$

i.e. the expected amount of time that X spends at y until time $T_{\text{mix}}^U(G)$ when started at x. For $A \subseteq V$, we set

$$g(x,A;G) = \sum_{y \in A} g(x,y;G).$$

We say that (G_n) is *uniformly locally transient* with *transience function* $\rho \colon [0,\infty) \times [0,\infty) \to [0,\infty)$ if

$$g(x,A;G_n) \leq \rho(d(x,A), \text{diam}(A)) \text{ for all } n, x \in V, \text{ and } A \subseteq V.$$

Here, $d(\cdot,\cdot)$ is the graph distance, $d(x,A) = \min_{y \in A} d(x,y)$, and $\rho(\cdot, s)$ is assumed to be non-increasing with $\lim_{r \to \infty} \rho(r,s) = 0$ when s is fixed. Let $\rho(r) = \rho(r,1)$,

$$\overline{\Delta}(G) = \max_{x \in V} \deg(x), \;\; \underline{\Delta}(G) = \min_{x \in V} \deg(x), \;\; \Delta(G) = \frac{\overline{\Delta}(G)}{\underline{\Delta}(G)}.$$

Assumption 1.1. *(G_n) is a sequence of uniformly locally transient graphs with $|V_n| \to \infty$ and such that there exists $\Delta_0 > 0$ so that $\Delta(G_n) \leq \Delta_0$ for all n and, for each $r > 0$,*

1. *$\log |B(x,r)| = o(\log |V_n|)$ as $n \to \infty$, and*
2. *$T_{\text{mix}}^U(G_n) \overline{\Delta}^r(G_n) = o(|V_n|)$ as $n \to \infty$.*

Assumption 1.2. *(G_n) is a sequence of graphs satisfying either*

1. *for every $\gamma > 0$ there exists $R_n^\gamma \to \infty$ as $n \to \infty$ satisfying $R_n^\gamma \leq \frac{1}{2} \max\{R > 0 : \max_{x \in V_n} |B(x,R)| \leq |V_n|^\gamma\}$ such that for every $r > 0$,*

$$\frac{T_{\text{mix}}^U(G_n)}{R_n^\gamma} \max_{d(x,A) \geq R_n^\gamma} g(x,A) = o(1) \text{ as } n \to \infty$$

uniformly $A \subseteq V_n$ with $\text{diam}(A) \leq r$, or

2. *a uniform Harnack inequality, i.e. for each $\alpha > 1$ there exists $C = C(\alpha) > 0$ such that for every $x, r, R > 0$ with $R/r \geq \alpha$ and positive harmonic function h on $B(x, R)$ we have that*

$$\max_{y \in B(x,r)} h(y) \leq C \min_{y \in B(x,r)} h(y).$$

THEOREM 1.3. *If (G_n) satisfies Assumptions 1.1 and 1.2 then for every $\epsilon > 0$,*

$$\lim_{n\to\infty} \|\mu(\cdot; \tfrac{1}{2} + \epsilon, G_n) - \nu(\cdot; G_n)\|_{TV} = 0 \text{ and}$$
$$\lim_{n\to\infty} \|\mu(\cdot; \tfrac{1}{2} - \epsilon, G_n) - \nu(\cdot; G_n)\|_{TV} = 1.$$

REMARK 1.4. *If (G_n) is a sequence with $|V_n| \to \infty$ and $\overline{\Delta}(G_n)$ is bounded in n, then Assumption 1.1 is equivalent to the decay of $g(x, y; G_n)$ in $d(x, y)$ uniformly in n.*

Many families of graphs satisfy Assumptions 1.1 and 1.2, for example $\mathbf{Z}_n^d$ for $d \geq 3$, random d-regular graphs whp, also for $d \geq 3$, and the hypercube $\mathbf{Z}_2^n$. We will discuss these and other examples in the next section.

The problem that we consider is closely related to determining the mixing time of the *lamplighter walk*, which we now introduce. If $G = (V, E)$ is a finite graph, the wreath product $G^\diamond = \mathbf{Z}_2 \wr G$ is the graph $(V^\diamond, E^\diamond)$ whose vertices are pairs (f, x) where $f \in \mathcal{X}(G)$ and $x \in V^\diamond$. There is an edge between (f, x) and (g, y) if and only if $\{x, y\} \in E$ and $f(z) = g(z)$ for $z \notin \{x, y\}$. $G^\diamond$ is also referred to as the *lamplighter graph* over G since it can be constructed by placing "lamps" at the vertices of G; the first coordinate f of a configuration (f, x) indicates the state of the lamps and the second gives the location of the lamplighter.

The *lamplighter walk* $X^\diamond$ on G is the random walk on $G^\diamond$. Its transition kernel $p(\cdot, \cdot; G^\diamond)$ can be constructed from $p(\cdot, \cdot; G)$ using the following procedure: given $(f, x) \in V^\diamond$,

1. sample $y \in V$ adjacent to x using $p(x, \cdot; G)$,
2. randomize the values of $f(x)$, $f(y)$ using independent fair coin flips,
3. move the lamplighter from x to y.

That both $f(x)$ and $f(y)$ are randomized rather than just $f(y)$ is necessary for reversibility.

Random walk on a sequence of graphs (G_n) is said to have a (total variation) cutoff with threshold (a_n) if

$$\lim_{n\to\infty} \frac{T_{\text{mix}}(\epsilon; G_n)}{a_n} = 1 \text{ for all } \epsilon > 0.$$

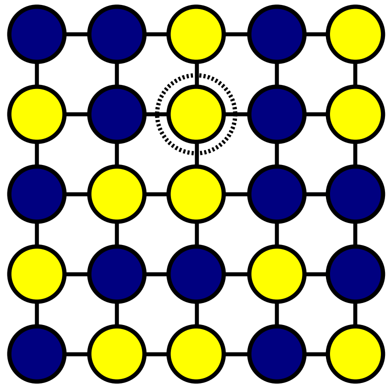


FIG 2. *A typical configuration of the lamplighter over a 5×5 planar grid. The colors indicate the state of the lamps and the dashed circle gives the position of the lamplighter.*

It is believed that many graphs have a cutoff, but establishing this is often quite difficult since it requires a delicate analysis of the behavior of the underlying random walk. The term was first coined by Aldous and Diaconis in [3] where they prove cutoff for the top-in-at-random shuffling process. Other early examples include random transpositions on the symmetric group [17], the riffle shuffle, and random walk on the hypercube [1]. By making a small modification to the proof of Theorems 1.3, we are able to establish cutoff for the lamplighter walk on base graphs satisfying Assumptions 1.1 and 1.2.

Before we state these results, we will first summarize previous work related to this problem. The mixing time of $G^\diamond$ was first studied by Häggström and Jonasson in [19] in the case $G_n = K_n$ and $G_n = \mathbf{Z}_n$. Their work implies a cutoff with threshold $\frac{1}{2}T_{\rm cov}(K_n)$ in the former case and that there is no cutoff in the latter. The connection between $T_{\rm mix}(G^\diamond)$ and $T_{\rm cov}(G)$ is explored further in [23], in addition to developing the relationship between the relaxation time of $G^\diamond$ and $T_{\rm hit}(G)$, and $\mathbf{E}2^{|\mathcal{L}(\alpha;G)|}$ and $T_{\rm mix}^U(G^\diamond)$. The results of [23] include a proof of cutoff when $G_n = \mathbf{Z}_n^2$ with threshold $T_{\rm cov}(\mathbf{Z}_n^2)$ and a general bound that

$$\left[\frac{1}{2} + o(1)\right] T_{\rm cov}(G_n) \leq T_{\rm mix}(G_n) \leq [1 + o(1)]T_{\rm cov}(G_n) \tag{1.1}$$

whenever (G_n) is a sequence of vertex transitive graphs with $T_{\rm hit}(G_n) = o(T_{\rm cov}(G_n))$. It is not possible to improve upon (1.1) without further hypotheses since the lower and upper bounds are acheived by K_n and $\mathbf{Z}_n^2$, respectively.

The bound (1.1) applies to $\mathbf{Z}_n^d$ when $d \geq 3$ since $T_{\rm hit}(\mathbf{Z}_n^d) \sim c_d n^d$ and

$T_{\rm cov}(\mathbf{Z}_n^d) = c_d' n^d (\log n)$ (see Proposition 10.13, Exercise 11.4 of [21]). This leads [23] to the question of whether there is a threshold for $T_{\rm mix}(\mathbf{Z}_n^d)$ and, if so, if it is at $\frac{1}{2}T_{\rm cov}(\mathbf{Z}_n^d)$, $T_{\rm cov}(\mathbf{Z}_n^d)$, or somewhere in between. By a slight extension of our methods, we are able to show that the threshold is at $\frac{1}{2}T_{\rm cov}(\mathbf{Z}_n^d)$ when $d \geq 3$, and that the same holds whenever (G_n) satisfies Assumptions 1.1 and 1.2.

THEOREM 1.5. *If (G_n) satisfies Assumptions 1.1 and 1.2 then $T_{\rm mix}(\epsilon; G_n^\diamond)$ has a cutoff with threshold $\frac{1}{2}T_{\rm cov}(G_n)$.*

In order to prove Theorems 1.3 and 1.5 we need to develop a delicate understanding of both the process of coverage and the correlation structure of $\mathcal{L}(\alpha; G_n)$. The proof yields the following theorem, which gives a precise estimate of correlation decay in $\mathcal{L}(\alpha; G_n)$ under the additional hypothesis of vertex transitivity.

THEOREM 1.6. *Suppose (G_n) is a sequence of vertex transitive graphs satisfying Assumption 1.1. If (x_n^i) for $1 \leq i \leq \ell$ is a family of sequences with $x_n^i \in V_n$ and $|x_n^i - x_n^j| \geq r$ for every n and $i \neq j$,*

$$(1.2) \qquad \begin{aligned}(1-\delta_{r,\ell})|V_n|^{-\ell\alpha - \delta_{r,\ell}} &\leq \mathbf{P}[x_n^i \in \mathcal{L}(\alpha; G_n) \text{ for all } i] \\ &\leq (1+\delta_{r,\ell})|V_n|^{-\ell\alpha + \delta_{r,\ell}}\end{aligned}$$

where $\delta_{r,\ell} \to 0$ as $r \to \infty$ while ℓ is fixed. If $\overline{\Delta}(G_n) \to \infty$ we take $r = 1$ and $\delta_{1,\ell} = o(1)$ as $n \to \infty$.

Outline. The remainder of the article is structured as follows. We show in Section 2 that the hypotheses of Theorems 1.3 and 1.5 hold for a number of natural examples. In Section 3, we collect several general estimates that will be used throughout the rest of the article; Proposition 3.2 is in particular of critical importance. Next, in Section 4 we will develop precise asymptotic estimates for the cover and hitting times of graphs (G_n) satisfying Assumption 1.1. The key idea is that the process of hitting a point can be understood by looking at the number $N(x,t)$ of excursions of X from $\partial B(x,r)$ to $\partial B(x,R)$ for $r < R$, then allowing the walk run for $\beta T_{\rm mix}^U(G)$ some $\beta > 0$ in order to remix. Uniform local transience implies that at the time x is hit, $N(x,t)$ is typically quite large and concentrated around its mean. This condition also gives that $\frac{1}{k}\sum_{j=1}^k p_j(x)$ is well concentrated around its mean, where $p_j(x)$ is the probability that the jth excursion hits x in time $\alpha T_{\rm mix}^U(G)$ after exiting $B(x,R)$, $\alpha \leq \beta$, conditional on its point of entry and the point of entry of the $(j+1)$st excursion. Decomposing the process of hitting into excursions

between concentric spheres is not new, and is used to great effect in [10], [11], [12], [13]. Our implementation of this idea is new since explicit representations of hitting probabilities and Green's functions in addition to the approximate rotational invariance in $\mathbf{Z}_n^d$ are simply not available in the generality we consider. We prove Theorem 1.2 in Section 5. This result, which may be of independent interest, is important in Section 6 since it allows us to deduce that points in $\mathcal{L}(\frac{1}{2}; G_n)$ are typically "spread apart." The article ends with a proof of Theorems 1.3 and 1.5 as well as a list of related open questions.

2. Examples.

$\mathbf{Z}_n^d$, $d \geq 3$. Although the simplest, this is the motivating example for this work. It is well-known (see Section 1.5 of [20]) that there exists a constant $c_d > 0$ so that $g(x, y; \mathbf{Z}_n^d) \leq c_d |x - y|^{2-d}$, which implies uniform local transience. Assumption 1.2 part (2) is also satisfied since it is also a basic result that random walk on $\mathbf{Z}_n^d$ satisfies a Harnack inequality (see [20], Section 1.4).

Super-critical percolation cluster. Suppose that η_e is a collection of iid random variables indexed by the edges $e = (x, y)$ of $\mathbf{Z}^d$, $d \geq 3$, taking values in $\{0, 1\}$ such that $\mathbf{P}[\eta_e = 1] = p \in [0, 1]$. An edge e is called open if $\eta_e = 1$. Let $\mathcal{C}(x)$ denote the subset of $\mathbf{Z}^d$ consisting of those elements y that can be connected to x by a path consisting only of open edges. Let C_∞ denote the event that there exists an infinite open cluster and let $p_c = \inf\{p > 0 : \mathbf{P}[C_\infty] > 0\}$. Suppose $p > p_c$. Then it is known that there exists a unique infinite open cluster $\mathcal{C}_\infty$ almost surely. Fix $x \in \mathcal{C}_\infty$ and consider the graph $G_n = B(x, n) \cap \mathcal{C}_\infty$. It follows from the works of Delmotte [9], Deuschel and Pisztora [15], Pisztora [24], and Benjamini and Mossel [6] that the heat kernel for continuous time random walk (CTRW) on G_n has Gaussian tails whp when n is large enough; see the discussion after the statement of Theorem A of [5]. Consequently, the Green's function of the CTRW on (G_n) has the same quantitative behavior as for $(\mathbf{Z}_n^d)$, which easily implies the same is true for the lazy random walk, which in turn yields uniform local transience for (G_n) whp when n is sufficiently large. Therefore there exists $n_0 = n_0(\omega)$ such that $(G_n : n \geq n_0(\omega))$ almost surely satisfies Assumption 1.1. Furthermore, it is a result of Barlow [5] that there exists $n_1 = n_1(\omega)$ such that random walk on $(G_n : n \geq n_1(\omega))$ almost surely satisfies a Harnack inequality and hence Assumption 1.2.

Bounded Degree Expanders. Suppose that (G_n) is an expander family with uniformly bounded maximal degree such that $|V_n| \to \infty$. Then there exists $T_0 < \infty$ such that $T_{\rm rel}(G_n) \le T_0$ for every n where $T_{\rm rel}(G_n)$ is the relaxation time of lazy random walk on G_n. Equation (12.11) of [21] implies that

$$p^t(x,y;G_n) \le C\left(\frac{1}{|V_n|} + e^{-t/T_0}\right)$$

and Theorem 12.3 of [21] gives $T_{\rm mix}^U(G_n) = O(\log|V_n|)$. By Remark 1.4, to check Assumption 1.1 we need only show the uniform decay $g(x,y;G_n)$ in $d(x,y)$. If $t < d(x,y)$, then it is obviously true that $p^t(x,y;G_n) = 0$. Hence (2.1)

$$g(x,y;G_n) \le C\left(\frac{O(\log|V_n|)}{|V_n|} + \sum_{t=d(x,y)}^{T_{\rm mix}^U(G_n)} e^{-t/T_0}\right) \le C_1 e^{-d(x,y)/T_0} + o(1)$$

as $n \to \infty$. We will now argue that (G_n) satisfies part (1) of Assumption 1.2. Suppose that $\overline{\Delta} \ge \max_{x\in V_n}\deg(x)$ for every n. We can obviously take $R_n^\gamma = \gamma\log|V_n|/(2\log\overline{\Delta})$, hence we have $T_{\rm mix}^U(G_n)/R_n^\gamma = O(1)$ as $n \to \infty$. Combining this with (2.1) implies that (G_n) satisfies Assumption 1.2.

Random Regular Graphs. Suppose that $d \ge 3$ and let $\mathcal{G}_{n,d}$ denote the set of d-regular graphs on n vertices. It is by now well-known [7] that, whp as $n \to \infty$, an element chosen uniformly from $\mathcal{G}_{n,d}$ is an expander. Consequently, whp, a sequence (G_n) where each G_n is chosen independently and uniformly from $\mathcal{G}_{n,d}$, $d \ge 3$, almost surely satisfies the hypotheses of our theorems.

Hypercube. As in the case of super-critical percolation, for $\mathbf{Z}_2^n$ it is easiest to prove bounds for the CTRW which, as we remarked before, easily translate over to the corresponding lazy walk. The transition kernel of the CTRW is

$$p^t(x,y;\mathbf{Z}_2^n) = \frac{1}{2^n}(1+e^{-2t/n})^{n-|x-y|}(1-e^{-2t/n})^{|x-y|},$$

where $|x-y|$ is the number of coordinates in which x and y differ. The spectral gap is $1/n$ (see Example 12.15 of [21]) which implies $\Omega(n) = T_{\rm mix}^U(\mathbf{Z}_2^n) = O(n^2)$ (see Theorem 12.3 of [21]). Suppose that $A \subseteq \mathbf{Z}_2^n$ has diameter s and $d(x,A) = r$. If $y \in A$, we have

$$p^t(x,y;\mathbf{Z}_2^n) \le \frac{1}{2^n}(1+e^{-2t/n})^{n-r}(1-e^{-2t/n})^r.$$

It is easy to see that

$$p^t(x,y;\mathbf{Z}_2^n) \leq \begin{cases} (C_\epsilon t/n)^r \exp[-(t/C_\epsilon n)(n-r)] \text{ if } t \leq \epsilon n, \\ e^{-\rho_\epsilon n} \text{ if } t > \epsilon n, \end{cases}$$

provided $\epsilon > 0$ is sufficiently small. Consequently,

$$g(x,A;\mathbf{Z}_2^n) \leq Cn^{s-r}$$

and therefore $\mathbf{Z}_2^n$ is uniformly locally transient. The other hypotheses of Assumption 1.1 are obviously satisfied. As for Assumption 1.2, we note that in this case, we can take $R_n^\gamma = \gamma n/(2\log_2 n)$. Thus if $r > 0$ it is easy to see that if $\text{diam}(A) \leq s$ and $d(x,A) \geq R_n^\gamma$ we have that

$$\sum_{y \in A} p^t(x,y;\mathbf{Z}_2^n) \leq n^s e^{-\rho_\epsilon n}.$$

if $t > \epsilon n$. On the other hand, if $t \leq \epsilon n$, then we have

$$\sum_{y \in A} p^t(x,y;\mathbf{Z}_2^n) \leq n^s \left(\frac{C_\epsilon t}{n}\right)^{\gamma n/(2\log_2 n)} e^{-t/2C_\epsilon}.$$

Hence it is not hard to see that $\mathbf{Z}_2^n$ satisfies Assumption 1.2.

Caley Graph of S_n Generated by Transpositions. Let G_n be the Caley graph of S_n generated by transpositions. By work of Diaconis and Shahshahani [17], $T_{\text{mix}}(G_n) = \Theta(n(\log n))$, which by Theorem 12.3 of [21] implies $T_{\text{mix}}^U(G_n) = O(n^2(\log n)^2)$. We are now going to give a crude estimate of $p^t(\sigma,\tau;S_n)$. By applying an automorphism, we may assume without loss of generality that $\sigma = \text{id}$. Suppose that $d(\text{id},\tau) = r$ and that $\tau_1,\ldots,\tau_r$ are transpositions such that $\tau_r \cdots \tau_1 = \tau$. Then $\tau_1,\ldots,\tau_r$ move at most $2r$ of the n elements of $\{1,\ldots,n\}$, say, $k_1,\ldots,k_{2r}$. Suppose $k_1',\ldots,k_{2r}'$ are distinct from $k_1,\ldots,k_{2r}$ and $\alpha \in S_n$ is such that $\alpha(k_i) = k_i'$ for $1 \leq i \leq r$. Then the automorphism of G_n induced by conjugation by α satisfies $\alpha\tau\alpha^{-1} \neq \tau$. Therefore the size of the set of elements τ' in S_n such that there exists a graph automorphism φ of G_n satisfying $\varphi(\tau) = \tau'$ and $\varphi(\text{id}) = \text{id}$ is at least $\binom{n}{2r} \geq 2^{-2r} n^{2r}((2r)!)^{-1}$ assuming $n \geq 4r$. Therefore

$$p^t(e,\tau;G_n) \leq \frac{2^{2r}(2r)!}{n^{2r}} \text{ and } g(e,\tau;G_n) \leq C(2^{2r}(2r)!)(\log n)^2 n^{2-2r}.$$

If $\text{diam}(A) = s$ then trivially $|A| \leq n^{2s}$ from which it is clear that (G_n) is uniformly locally transient. The other parts of Assumption 1.1 are obviously

satisfied by G_n. As for Assumption 1.2, a simple calculation shows that we can take $R_n^\gamma \leq \gamma n/4 + O(1)$. Hence setting $R_n^\gamma = \sqrt{n}$, a calculation analogous to the one above gives that Assumption 1.2 is satisfied.

3. Preliminary Estimates. The purpose of this section is to collect several general estimates that will be useful for us throughout the rest of the article.

LEMMA 3.1. *If μ, ν are measures with ν absolutely continuous with respect to μ and*

$$\int \frac{d\nu}{d\mu} d\nu = 1 + \epsilon$$

then

$$\|\nu - \mu\|_{TV} \leq \frac{\sqrt{\epsilon}}{2}.$$

PROOF. This is a consequence of the Cauchy-Schwarz inequality:

$$\|\mu - \nu\|_{TV}^2 = \left(\frac{1}{2}\int \left|\frac{d\nu}{d\mu} - 1\right| d\mu\right)^2 \leq \frac{1}{4}\int \left|\frac{d\nu}{d\mu} - 1\right|^2 d\mu = \frac{1}{4}\left(\int \frac{d\nu}{d\mu} d\nu - 1\right)$$

$\square$

Let ν denote the uniform measure on $\mathcal{X}(G) = \{f \colon V \to \{0,1\}\}$

PROPOSITION 3.2. *Suppose that μ is a measure on $\mathcal{X}(G)$ given by first sampling $\mathcal{R} \subseteq V$ according to a probability μ_0 on $2^{|V|}$, then, conditional on $\mathcal{R}$ sampling $f \in \mathcal{X}(G)$ by setting*

$$f(x) = \begin{cases} \xi(x) \text{ if } x \in \mathcal{R}, \\ 0 \text{ otherwise,} \end{cases}$$

where $(\xi(x) : x \in V)$ is a collection of iid random variables with $\mathbf{P}[\xi(x) = 0] = \mathbf{P}[\xi(x) = 1] = \frac{1}{2}$. Then

$$\int \frac{d\mu}{d\nu} d\mu = \int \int 2^{|\mathcal{R}^c \cap \mathcal{S}^c|} d\mu_0(\mathcal{R}) d\mu_0(\mathcal{S}).$$

PROOF. Suppose $f \in \mathcal{X}(G)$ is such that $f|\mathcal{R}^c \equiv 0$ for some $\mathcal{R} \subseteq V$. Letting $\mu(\cdot|\mathcal{S})$ be the conditional law of μ given $\mathcal{S}$, we have

$$\begin{aligned} \int \frac{d\mu}{d\nu} d\mu &= 2^N \int \mu(\{f\}) d\mu(f) \\ &= 2^N \int \int \left(\int \mu(\{f\}|\mathcal{S}) d\mu_0(\mathcal{S}) \right) d\mu(f|\mathcal{R}) d\mu_0(\mathcal{R}). \end{aligned}$$

Note that

$$\mu(\{f\}|\mathcal{S}) = 2^{-|\mathcal{R}\cap\mathcal{S}|-|\mathcal{S}\setminus\mathcal{R}|}|\mathbf{1}_{\{f|\mathcal{R}\setminus\mathcal{S}\equiv 0\}}.$$

Hence, the above is equal to

$$\begin{aligned}
&2^N \int\int \left(\int 2^{-|\mathcal{R}\cap\mathcal{S}|-|\mathcal{S}\setminus\mathcal{R}|}\mathbf{1}_{\{f|\mathcal{R}\setminus\mathcal{S}\equiv 0\}} d\mu_0(\mathcal{S})\right) d\mu(f|\mathcal{R}) d\mu_0(\mathcal{R})\\
=&2^N \int\int 2^{-|\mathcal{R}\cap\mathcal{S}|-|\mathcal{S}\setminus\mathcal{R}|} \left(\int \mathbf{1}_{\{f|\mathcal{R}\setminus\mathcal{S}\equiv 0\}} d\mu(f|\mathcal{R})\right) d\mu_0(\mathcal{R}) d\mu_0(\mathcal{S})\\
=&2^N \int\int 2^{-|\mathcal{R}\cap\mathcal{S}|-|\mathcal{S}\setminus\mathcal{R}|} 2^{-|\mathcal{R}\setminus\mathcal{S}|} d\mu_0(\mathcal{S}) d\mu_0(\mathcal{R}),
\end{aligned}$$

where $N = |V|$. Simplifying the expression in the exponent gives the result. □

Roughly speaking, the general strategy of our proof will be to show that if $\mathcal{R}, \mathcal{R}'$ denote independent copies of the range of random walk on G_n run up to time $(\frac{1}{2}+\epsilon)T_{\text{cov}}(G_n)$ and $\mathcal{L} = V \setminus \mathcal{R}$, $\mathcal{L}' = V \setminus \mathcal{R}'$ then

$$\mathbf{E}\exp(\zeta|\mathcal{L}\cap\mathcal{L}'|) = 1 + o(1) \text{ as } n \to \infty \tag{3.1}$$

for $\zeta > 0$. This method cannot be applied directly, however, since this exponential moment blows up even in the case of $\mathbf{Z}_n^3$. To see this, suppose that X, X' are independent random walks on $\mathbf{Z}_n^3$ initialized at stationarity. We divide the cover time $c_3 n^3(\log n)$ into rounds of length n^2. In the first round, with probability $1/4$ we know that X starts in $\mathbf{L}_1 = \mathbf{Z}_n^2 \times \{n/8, \ldots, 3n/8\}$. In each successive round, X has probability $\rho_0 > 0$ strictly bounded from zero in n of not leaving $\mathbf{L}_2 = \mathbf{Z}_n^2 \times \{1, \ldots, n/2\}$ and ending the round in $\mathbf{L}_1$. Since there are $c_3 n(\log n)$ rounds, this means that X does not leave $\mathbf{L}_1$ with probability at least

$$\tfrac{1}{4}\rho_0^{c_3 n \log n} \geq c\exp(-\rho_1 n \log n).$$

Since X' satisfies the same estimate, we therefore have

$$\mathbf{E}\exp(\zeta|\mathcal{L}\cap\mathcal{L}'|) \geq c\exp(\tfrac{\zeta}{2}n^3 - 2\rho_1 n\log n) \to \infty \text{ as } n \to \infty.$$

The idea of the proof is to truncate the exponential moment in (3.1) by replacing μ_0 with $\widetilde{\mu}_0$, the law of $\mathcal{R}(\frac{1}{2}+\delta; G_n)$ *conditional on typical behavior* so that

$$\|\widetilde{\mu}_0 - \mu_0\|_{TV} = o(1) \text{ as } n \to \infty.$$

We do this in such a way that the uncovered set exhibits a great deal of spatial independence in order to make the exponential moment easy to estimate. To this end, we will condition on two different events. The first is

that points in $\mathcal{L}(\frac{1}{2}+\delta;G_n)$ are well-separated: for any $x \in V_n$ we have $|\mathcal{L}(\frac{1}{2}+\delta;G_n) \cap B(x,R_n^\gamma)| \le M = M_{\gamma,\delta}$. Given this event, we can partition $\mathcal{L}(\frac{1}{2}+\delta;G_n)$ into disjoint subsets $E_1,\ldots,E_M$ such that $x,y \in E_\ell$ distinct implies $d(x,y) \ge R_n^\gamma$. Observe:

$$\mathbf{E}\exp(\zeta|\mathcal{L}\cap\mathcal{L}'\cap E_\ell|) \le \mathbf{E}\prod_{x\in E_\ell}\left(1+e^\zeta \prod_{j=1}^{N'(x,T)}(1-q_j'(x))\right).$$

where $N'(x,T)$ is the number of excursions of X' from $\partial B(x,r)$ to $\partial B(x,R)$ by time T and $q_j'(x)$ is the probability the jth such excursion hits x conditional on its entrance and exit points. When T is large, uniform local transience implies that $N'(x,T)$ and $\prod_{j=1}^k q_j'(x)$ can be estimated by their mean and, roughly speaking, this is the second event on which we will condition. Finally, we get control of the entire exponential moment by an application of Hölder's inequality.

We finish the section by recording a standard lemma that bounds the rate of decay of the total variation and uniform distances to stationarity:

PROPOSITION 3.3. *For every $s,t \in N$,*

$$\max_x \|p^{t+s}(x,\cdot)-\pi\|_{TV} \le 4\max_{x,y}\|p^t(x,\cdot)-\pi\|_{TV}\|p^s(y,\cdot)-\pi\|_{TV} \tag{3.2}$$

$$\max_{x,y}\left|\frac{p^{t+s}(x,y)}{\pi(y)}-1\right| \le \max_{x,y}\frac{p^s(x,y)}{\pi(y)}\max_x\|p^t(x,\cdot)-\pi\|_{TV}. \tag{3.3}$$

PROOF. The first part is a standard result; see, for example, Lemmas 4.11 and 4.12 of [21]. The second part is a consequence of the semigroup property:

$$\begin{aligned}\frac{1}{\pi(z)}p^{t+s}(x,z) &= \frac{1}{\pi(z)}\sum_y p^t(x,y)p^s(y,z)\\ &= \frac{1}{\pi(z)}\sum_y[p^t(x,y)-\pi(y)+\pi(y)]p^s(y,z)\\ &\le \left(\max_{y,z}\frac{p^s(y,z)}{\pi(z)}\right)\|p^t(x,\cdot)-\pi\|_{TV}+1\end{aligned}$$

□

Note that (3.2) and (3.3) give

$$\max_x\|p^t(x,\cdot)-\pi\|_{TV} \le ce^{-c\alpha} \text{ for } t \ge \alpha T_{\text{mix}}(G) \tag{3.4}$$

$$\max_{x,y}\left|\frac{p^{t+s}(x,y)}{\pi(y)}-1\right| \le ce^{-c\alpha} \text{ for } t \ge T^U_{\text{mix}}(G)+\alpha T_{\text{mix}}(G) \tag{3.5}$$

where $c > 0$ is a universal constant. We will often use (3.5) without reference, and, for simplicity use that the same inequality holds when $T_{\text{mix}}^U(G) + \alpha T_{\text{mix}}(G)$ is replaced by $\alpha T_{\text{mix}}^U(G)$, perhaps adjusting $c > 0$.

4. Hitting and Cover Times. In this section we will develop a number of estimates useful for understanding the process of coverage via excursions of random walk between concentric spheres. Throughout, we assume that we have a sequence (G_n) satisfying Assumption 1.1 with transience function ρ. We will often suppress the index n and refer to an element of (G_n) as G.

4.1. *Probability of Success.* Fix $R > r$ and let X be a lazy random walk on G. Suppose $E = \{x_1, \ldots, x_\ell\} \subseteq V$ where $d(x_i, x_j) \geq 2R$ for $i \neq j$. Let $E(s) = \{x \in V : d(x, E) \leq s\}$ where $d(x, E) = \min_{y \in E} d(x, y)$.

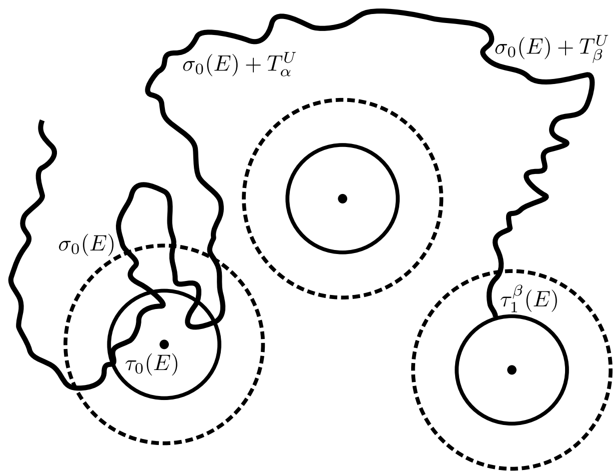


Fig 3. *The solid and dashed circles represent the boundaries of $E(r)$ and $E(R)$, respectively, and the small points are the elements of E. Note that X may re-enter $E(r)$ during the interval $[\sigma_k^\beta(E), \tau_{k+1}^\beta(E)]$*

.

Fix $\beta \geq 0$ and define stopping times

$$\tau_0(E) = \min\{t \geq 0 : X(t) \in \partial E(r)\},$$
$$\sigma_0(E) = \min\{t \geq \tau_0(E) : X(t) \notin E(R)\}$$

and inductively define

$$\tau_k^\beta(E) = \min\{t \geq \sigma_{k-1}^\beta(E) + T_\beta^U : X(t) \in \partial E(r)\},$$
$$\sigma_k^\beta(E) = \min\{t \geq \tau_k^\beta(E) : X(t) \notin E(R)\},$$

where $\partial E(r) = \{z : d(z, E) = r\}$.

Fix $\alpha \in [0, \beta]$. Let $S_j^{\alpha,\beta}(x; E)$ be the event that $X(t)$ hits x in $[\tau_j^\beta(E), \sigma_j^\beta(E) + T_\alpha^U]$,

$$p_j^{\alpha,\beta}(x; E) = \mathbf{P}[S_j^{\alpha,\beta}(x; E) | X(\tau_j^\beta(E)), X(\tau_{j+1}^\beta(E))],$$

and

$$a_j^{\alpha,\beta}(x; E) = \mathbf{E}\left[\sum_{t=\tau_j^\beta(E)}^{\sigma_j^\beta(E)+T_\alpha^U} \mathbf{1}_{\{X(t)=x\}} \middle| X(\tau_j^\beta(E)), X(\tau_{j+1}^\beta(E))\right].$$

Finally, let $\overline{p}_{r,R}^{\alpha,\beta}(x; E) = \mathbf{E}_\pi p_0^{\alpha,\beta}(x; E)$ and $\overline{a}_{r,R}^{\alpha,\beta}(x; E) = \mathbf{E}_\pi a_0^{\alpha,\beta}(x; E)$. For $\beta \geq \alpha \geq 1$ note that

$$\overline{p}_{r,R}^{\alpha,\beta}(x; E) = \overline{p}_{r,R}^{1,1}(x; E) + O\left(\frac{T_\beta^U}{|V|}\right).$$

Since $\overline{p}_{r,R}^{1,1}(x; E) \geq c\overline{\Delta}^{-r}(G)$ and $T_\beta^U \overline{\Delta}^r(G)/|V| = o(1)$ as $n \to \infty$, we therefore have

$$\overline{p}_{r,R}^{\alpha,\beta}(x; E) = (1 + o(1))\overline{p}_{r,R}^{1,1}(x; E). \tag{4.1}$$

From now on, we will write $\overline{p}_{r,R}(x; E)$ for $\overline{p}_{r,R}^{1,1}(x; E)$. It is also true that $\overline{a}_{r,R}^{\alpha,\beta}(x; E) = (1+o(1))\overline{a}_{r,R}^{1,1}(x; E)$ and we will also write $\overline{a}_{r,R}(x; E)$ for $\overline{a}_{r,R}^{1,1}(x; E)$.

Lemma 4.1. *For each $\delta > 0$ there exists $\gamma_0 > 0$ such that for $\beta - \alpha \geq \gamma_0$ and all n large enough we have*

$$1 - \delta \leq \frac{p_j^{\alpha,\beta}(x; E)}{\mathbf{P}[S_j^{\alpha,\beta}(x) | X(\tau_j^\beta(E))]} \leq 1 + \delta, \tag{4.2}$$

$$1 - \delta \leq \frac{a_j^{\alpha,\beta}(x; E)}{\mathbf{E}\left[\sum_{t=\tau_j^\beta(E)}^{\sigma_j^\beta(E)+T_\alpha^U} \mathbf{1}_{\{X(t)=x\}} \middle| X(\tau_j^\beta(E))\right]} \leq 1 + \delta. \tag{4.3}$$

In particular, $p_j^{\alpha,\beta}(x; E) \leq (1 + \delta)\rho(r)$ and $a_j^\beta(x; E) \leq (1 + \delta)\rho(0)\rho(r)$.

Proof. Note that

$$\begin{aligned}
&\mathbf{P}[X(\sigma_j^\beta(E)+T_\alpha^U)=z|X(\tau_j^\beta(E))=z_j, X(\tau_{j+1}^\beta(E))=z_{j+1}]\\
=&\frac{\mathbf{P}[X(\sigma_j^\beta(E)+T_\alpha^U)=z, X(\tau_j^\beta(E))=z_j, X(\tau_{j+1}^\beta(E))=z_{j+1}]}{\mathbf{P}[X(\tau_j^\beta(E))=z_j, X(\tau_{j+1}^\beta(x;E))=z_{j+1}]}\\
=&\left(\frac{\mathbf{P}[X(\tau_{j+1}^\beta(E))=z_{j+1}|X(\sigma_j^\beta(E)+T_\alpha^U)=z]}{\mathbf{P}[X(\tau_{j+1}^\beta(E))=z_{j+1}|X(\tau_j^\beta(E))=z_j]}\right)\\
&\mathbf{P}[X(\sigma_j^\beta(E)+T_\alpha^U)=z|X(\tau_j^\beta(E))=z_j].
\end{aligned}$$

Mixing considerations imply

$$\mathbf{P}[X(\tau_{j+1}^\beta(E))=z_{j+1}|X(\tau_j^\beta(E))=z_j]=\big[1+O(e^{-c\beta})\big]\mathbf{P}_\pi[X(\tau_0(E))=z_{j+1}],$$

and

$$\begin{aligned}
&\mathbf{P}[X(\tau_{j+1}^\beta(E))=z_{j+1}|X(\sigma_j^\beta(E)+T_\alpha^U)=z]\\
=&\big[1+O(e^{-c(\beta-\alpha)})\big]\mathbf{P}_\pi[X(\tau_0(E))=z_{j+1}].
\end{aligned}$$

Consequently, if μ_j denotes the law of $X(\sigma_j^\beta(E)+T_\alpha^U)$ conditional on $X(\tau_j^\beta(E))$ and $X(\tau_{j+1}^\beta(E))$ and μ is the law of $X(\sigma_j^\beta(E)+T_\alpha^U)$ but conditional only on $X(\tau_j^\beta(E))$, we have $1-\delta\le d\mu_j/d\mu\le 1+\delta$ when $\beta-\alpha$ is large enough. Thus,

$$\begin{aligned}
p_j^{\alpha,\beta}(x;E)=&\int \mathbf{P}[S_j^{\alpha,\beta}(x)|X(\tau_j^\beta(E)),X(\sigma_j^\beta(E)+T_\alpha^U)=z,X(\tau_{j+1}^\beta(E))]d\mu_j(z)\\
\le&(1+\delta)\int \mathbf{P}[S_j^{\alpha,\beta}(x)|X(\tau_j^\beta(E)),X(\sigma_j^\beta(E)+T_\alpha^U)=z]d\mu(z)\\
=&(1+\delta)\mathbf{P}[S_j^{\alpha,\beta}(x)|X(\tau_j^\beta(E))]
\end{aligned}$$

The lower bound for $p_j^{\alpha,\beta}(x;E)$ and the bounds for $a_j(x;E)$ are proved similarly. □

Lemma 4.2. *Fix $r>0$ and $\delta\in(0,1)$. There exists $\gamma_0>0$ depending only on r,δ such that for all $R\ge r$, $\beta-\alpha\ge\gamma_0$, and n large enough we have*

$$\begin{aligned}
\mathbf{P}&\left[\sum_{j=1}^k p_j^{\alpha,\beta}(x;E)\notin[1-\delta,1+\delta]\overline{p}_{r,R}(x;E)k\right]\\
&\le 4\exp\left(-\frac{C\delta^2\overline{p}_{r,R}(x;E)}{\rho(r)}k\right)
\end{aligned}\tag{4.4}$$

and

$$\mathbf{P}\left[\sum_{j=1}^{k} a_j^{\alpha,\beta}(x;E) \notin [1-\delta, 1+\delta]\overline{a}_{r,R}(x;E)k\right] \leq 4\exp\left(-\frac{C\delta^2 \overline{a}_{r,R}(x;E)}{\rho(r)}k\right) \tag{4.5}$$

where $C > 0$ *is independent of* r, R, δ.

Proof. Let μ be the measure on $\partial E(r)$ induced by the law of $X(\tau_0(E))$ given that X has a stationary initial distribution. For each $\delta > 0$, let $\mathcal{M}(\delta)$ be the set of measures ν on $\partial E(r)$ satisfying

$$\max_{z \in \partial E(r)} \left|\frac{\nu(z)}{\mu(z)} - 1\right| + \max_{z \in \partial E(r)} \left|\frac{\mu(z)}{\nu(z)} - 1\right| \leq \delta. \tag{4.6}$$

Let $\mu_y(z) = \mathbf{P}_y[X(\tau^\gamma(E)) = z]$ where $\tau^\gamma(E) = \min\{t \geq T_\gamma^U : X(t) \in \partial E(r)\}$. Mixing considerations imply that $\mu_y \in \mathcal{M}(Ce^{-C\gamma})$ for some $C > 0$. Fix $\delta > 0$, $\delta' < \delta/2$, and take $\beta - \alpha = \gamma$ so large that $Ce^{-C\gamma} \leq \delta'/2$. Let $\overline{\mu}$, $\underline{\mu}$ be elements of $\mathcal{M}(\delta'/2)$ such that $\mathbf{P}[S_0^{\alpha,\beta}(x) | X(\tau_0(E)) = Z]$ where $Z \sim \overline{\mu}, \underline{\mu}$ stochastically dominates from above and below, respectively, all other choices in $\mathcal{M}(\delta'/2)$. Assume that γ_0 is chosen sufficiently large so that the previous lemma applies for $\delta'/2$ when n is sufficiently large.

Let $(U_j), (L_j)$ be iid sequences with laws $\mathbf{P}[S_0^{\alpha,\beta}(x) | X(\tau_0(E)) = Z]$, $Z \sim \overline{\mu}, \underline{\mu}$, respectively. With $\overline{U} = \mathbf{E}U_1$ and $\overline{L} = \mathbf{E}U$, obviously

$$(1-\delta')\overline{p}_{r,R}(x;E) \leq \overline{L} \leq \overline{U} \leq (1+\delta')\overline{p}_{r,R}(x;E).$$

By construction, we can find a coupling of $U_j, L_j, p_j^{\alpha,\beta}(x;E)$ so that

$$L_j \leq p_j^{\alpha,\beta}(x;E) \leq U_j \text{ for all } j.$$

Corollary 2.4.5 of [14] implies

$$\mathbf{E}e^{\lambda U_1} \leq \frac{1}{2\rho(r)}(\overline{U}e^{2\lambda\rho(r)} + 2\rho(r) - \overline{U})$$

hence Exercise 2.2.26 of [14] gives that the Fenchel-Legendre transform Λ^* of the law of U_1 satisfies

$$\Lambda^*(u) \geq \widetilde{\Lambda}^*(u) \equiv \frac{u}{2\rho(r)}\log\left(\frac{u}{\overline{U}}\right) + \left(1 - \frac{u}{2\rho(r)}\right)\log\left(\frac{1-u/(2\rho(r))}{1-\overline{U}/(2\rho(r))}\right).$$

As

$$\widetilde{\Lambda}^*(\overline{U}) = (\widetilde{\Lambda}^*)'(\overline{U}) = 0 \text{ and } (\widetilde{\Lambda}^*)''(u) \geq \frac{1}{2\rho(r)u}$$

we have

$$\inf_{u \geq (1+\delta')\overline{U}} \Lambda^*(u) \geq \frac{1}{4\rho(r)\overline{U}}(\delta')^2\overline{U}^2 = \frac{(\delta')^2\overline{U}}{4\rho(r)},$$

assuming $\delta' < 1$. Consequently, Cramer's theorem (Theorem 2.2.3, part (c), of [14]) implies that

$$\mathbf{P}\left[\sum_{i=1}^k U_i \leq (1+\delta')\overline{U}k\right] \geq 1 - 2\exp\left(-\frac{(\delta')^2\overline{U}k}{4\rho(r)}\right). \tag{4.7}$$

An analogous estimate also holds for (L_i) with $\overline{U}$ replaced by $\overline{L}$. The proof of concentration for the $a_j^{\alpha,\beta}(x;E)$ is the same. □

4.2. *Excursion Lengths.* In this subsection we will estimate the mean and prove concentration for the lengths $\tau_{k+1}^\beta(E) - \tau_k^\beta(E)$ of successive excursions. Before we do this, it will be helpful for us to estimate the Radon-Nikodym derivative of the law of random walk conditioned not to hit $E(r)$ with respect to the stationary measure π.

LEMMA 4.3. *For $\alpha, s \geq 0$ we have*

$$\mathbf{P}_y[X(T_\alpha^U) = z|\mathcal{A}] = \left[1 + O(e^{-c\alpha} + |E|\rho(s,r)) + o(1)\right]\pi(z) \text{ as } n \to \infty$$

where $\mathcal{A} = \{\tau(E) \geq T_\alpha^U, d(X(T_\alpha^U), E) \geq s\}$.

PROOF. For $z \in V$ with $d(z,E) \geq s$, observe

$$\begin{aligned}\mathbf{P}_y[X(T_\alpha^U) = z|\mathcal{A}] &= \frac{\mathbf{P}_y[X(T_\alpha^U) = z, \tau(E) \geq T_\alpha^U]}{\mathbf{P}_y[\mathcal{A}]}\\ &= \frac{\mathbf{P}_y[\tau(E) \geq T_\alpha^U | X(T_\alpha^U) = z]\mathbf{P}_y[X(T_\alpha^U) = z]}{\mathbf{P}_y[\mathcal{A}]}\\ &= (1 + O(e^{-c\alpha}))\frac{\mathbf{P}_y[\tau(E) \geq T_\alpha^U | X(T_\alpha^U) = z]\pi(z)}{\mathbf{P}_y[\mathcal{A}]}.\end{aligned}$$

For $\alpha' < \alpha$,

$$\begin{aligned}&\mathbf{P}_y[\tau(E) \geq T_\alpha^U | X(T_\alpha^U) = z]\\ =&\mathbf{P}_y[\tau(E) \geq T_\alpha^U - T_{\alpha'}^U | X(T_\alpha^U) = z] - \mathbf{P}_y[T_\alpha^U > \tau(E) \geq T_\alpha^U - T_{\alpha'}^U | X(T_\alpha^U) = z]\end{aligned}$$

We have

$$\mathbf{P}_y[\tau(E) \geq T_\alpha^U - T_{\alpha'}^U | X(T_\alpha^U) = z] = 1 - \frac{\mathbf{P}_y[\tau(E) < T_\alpha^U - T_{\alpha'}^U, X(T_\alpha^U) = z]}{\mathbf{P}_y[X(T_\alpha^U) = z]}$$
$$= 1 - \frac{1 + O(e^{-c\alpha})}{\pi(z)} \mathbf{P}_y[X(T_\alpha^U) = z | \tau(E) < T_\alpha^U - T_{\alpha'}^U] \mathbf{P}_y[\tau(E) < T_\alpha^U - T_{\alpha'}^U]$$
$$= \mathbf{P}_y[\tau(E) \geq T_\alpha^U - T_{\alpha'}^U] + O(e^{-c(\alpha - \alpha')}).$$

Note that

$$\mathbf{P}_y[T_\alpha^U > \tau(E) \geq T_\alpha^U - T_{\alpha'}^U | X(T_\alpha^U) = z]$$
$$= \frac{1 + O(e^{-c\alpha})}{\pi(y)\pi(z)} \mathbf{P}_y[T_\alpha^U > \tau(E) \geq T_\alpha^U - T_{\alpha'}^U, X(T_\alpha^U) = z] \pi(y).$$

By reversibility, this is equal to

$$\frac{1 + O(e^{-c\alpha})}{\pi(y)} \mathbf{P}_z[\tau(E) \leq T_{\alpha'}^U, d(X(t), E) > r \text{ for all } T_{\alpha'}^U < t \leq T_\alpha^U, X(T_\alpha^U) = y]$$
$$\leq \frac{1 + O(e^{-c\alpha})}{\pi(y)} \mathbf{P}_z[X(T_\alpha^U) = y | \tau(E) \leq T_{\alpha'}^U] \mathbf{P}_z[\tau(E) \leq T_{\alpha'}^U]$$
$$= (1 + O(e^{-c(\alpha - \alpha')})) \mathbf{P}_z[\tau(E) \leq T_{\alpha'}^U]$$

A union bound implies this is of order $O\left(|E|\rho(s, r) + o(1)\right)$. With $\mathcal{A}_1 = \{d(X(T_\alpha^U), E) \geq s\}$,

$$\mathbf{P}_y[\mathcal{A}] = \mathbf{P}_y[\tau(E) \geq T_\alpha^U, \mathcal{A}_1]$$
$$= (\mathbf{P}_y[\tau(E) \geq T_\alpha^U - T_{\alpha'}^U | \mathcal{A}_1] - \mathbf{P}_y[T_\alpha^U > \tau(E) \geq T_\alpha^U - T_{\alpha'}^U | \mathcal{A}_1]) \mathbf{P}_y[\mathcal{A}_1]$$
$$= \mathbf{P}_y[\tau(E) \geq T_\alpha^U - T_{\alpha'}^U] + O\left(e^{-c(\alpha - \alpha')} + |E|\rho(s, r) + o(1)\right),$$

the last line coming from a similar analysis as before. Consequently,

$$\frac{\mathbf{P}_y[\tau(E) \geq T_\alpha^U | X(T_\alpha^U) = z]}{\mathbf{P}_y[\mathcal{A}]} = 1 + O\left(e^{-c(\alpha - \alpha')} + |E|\rho(s, r) + o(1)\right).$$

Taking $\alpha' = \alpha/2$ gives the lemma. □

Let $\tau_k(E) = \tau_k^0(E)$, $\sigma_k(E) = \sigma_k^0(E)$, and $T_{r,R}(E) = \mathbf{E}_\pi[\tau_1(E) - \tau_0(E)]$.

Lemma 4.4 (Mean Excursion Length). *For every $r, \delta > 0$ there exists $R_0 > r$ such that $R \geq R_0$ implies*

$$(1 - \delta) T_{r,R}(E) \leq \min_{y \notin E(R)} \mathbf{E}_y \tau_0(E) \leq \max_{y \notin E(R)} \mathbf{E}_y \tau_0(E) \leq (1 + \delta) T_{r,R}(E)$$

for all n large enough.

Proof. We have that

$$\mathbf{E}_\pi[\tau_1(E)-\tau_0(E)] = \mathbf{E}_\pi[\sigma_0(E)-\tau_0(E)] + \mathbf{E}_\pi[\tau_1(E)-\sigma_0(E)].$$

Obviously,

$$\mathbf{E}_\pi[\sigma_0(E)-\tau_0(E)] \leq \max_{y\in E(r)} \mathbf{E}_y \sigma_0(E) \leq cT_{\mathrm{mix}}^U(G)$$

for some $c>0$ since in each interval of length $T_{\mathrm{mix}}^U(G)$, random walk started in $E(r)$ has probability uniformly bounded from below of leaving $E(R)$ provided n is large enough. It is also obvious that

$$\min_{y\notin E(R)} \mathbf{E}_y \tau_0(E) \leq \mathbf{E}_\pi[\tau_1(E)-\sigma_0(E)] \leq \max_{y\notin E(R)} \mathbf{E}_y\tau_0(E).$$

The previous lemma implies

$$(1-\delta)\mathbf{E}_\pi[\tau_0(E)] \leq \mathbf{E}_y[\tau_0(E)|\mathcal{A}] \leq T_\alpha^U + (1+\delta)\mathbf{E}_\pi[\tau_0(E)]$$

for all $y \notin E(R)$ provided we choose R,α,s,n large enough to accommodate our choice of δ. Hence,

$$(1-\delta)\mathbf{E}_\pi[\tau_0(E)] \leq \mathbf{E}_y[\tau_0(E)] \leq (1+\delta)\mathbf{E}_\pi[\tau_0(E)],$$

as it is not difficult to see that $T_{\mathrm{mix}}^U(G) = o(T_{r,R}(E))$ as $n\to\infty$. Therefore

$$\max_{y_1,y_2\notin E(R)} \frac{\mathbf{E}_{y_1}\tau_0(E)}{\mathbf{E}_{y_2}\tau_0(E)} \leq 1+\delta,$$

which proves the lemma. $\square$

Lemma 4.5 (Concentration of Excursions). *For each $\beta \geq 0$ and $r,\delta>0$ there exists $R_0 > r$ such that*

$$\mathbf{P}_y\left[\tau_k^\beta(E) \leq (1-\delta)T_{r,R}(E)k\right] \leq e^{-C\delta^2 k} \tag{4.8}$$

$$\mathbf{P}_y\left[\tau_k^\beta(E) \geq (1+\delta)T_{r,R}(E)k\right] \leq e^{-C\delta^2 k} \tag{4.9}$$

for all $R \geq R_0$, $y\in V$, and n large enough.

Proof. We adapt the proof of Lemma 2.4 of [10] to our setting. First of all, it follows from Lemma 4.4 that

$$\max_y \mathbf{E}_y[\tau_0(E)] \leq CT_{r,R}(E)$$

for some $C > 0$ provided R, n are sufficiently large. Consequently, Kac's moment formula (see [18], equation 6) for the first hitting time of a strong Markov process implies for any $j \in \mathbf{N}$ we have that

$$\max_y \mathbf{E}_y[(\tau_0(E))^j] \leq j! c^j T^j_{r,R}(E) \tag{4.10}$$

for some $c > 0$. This implies that there exists $\lambda_0 > 0$ so that

$$\max_y \mathbf{E}_y \exp[\lambda \tau_0(E)/T_{r,R}(E)] < \infty \text{ for all } \lambda \in (0, \lambda_0).$$

Using $\mathbf{E}[\sigma_0(E) - \tau_0(E)] = o(T_{r,R}(E))$, a similar argument implies that, by possibly decreasing λ_0,

$$\max_y \mathbf{E}_y \exp[\lambda \sigma_0(E)/T_{r,R}(E)] < \infty \text{ for all } \lambda \in (0, \lambda_0).$$

Combining the strong Markov property with $T^U_\beta = o(T_{r,R}(E))$ yields

$$\max_y \mathbf{E}_y \exp[\lambda \tau^\beta_k(E)/T_{r,R}(E)] < \infty \text{ for all } \lambda \in (0, \lambda_0).$$

Let R_0 be large enough so that the previous lemma implies

$$(1 - \delta/2) T_{r,R}(E) \leq \min_{y \notin E(R)} \mathbf{E}_y \tau_0(E) \leq \max_{y \notin E(R)} \mathbf{E}_y \tau_0(E) \leq (1 + \delta/2) T_{r,R}(E)$$

for $R \geq R_0$ and n large enough. We compute,

$$\begin{aligned} &\max_{y \notin E(R)} \mathbf{E}_y e^{-\theta \tau_0(E)} \leq 1 - \theta \min_{y \notin E(R)} \mathbf{E}_y \tau_0(E) + \theta^2 \max_{y \notin E(R)} \mathbf{E}_y \tau_0^2(E) \\ &\leq 1 - \theta(1 - \delta/2) T_{r,R}(E) + \rho\theta^2 \leq \exp(\rho\theta^2 - \theta(1 - \delta/2) T_{r,R}(E)). \end{aligned}$$

where $\rho = c T^2_{r,R}(E)$ for some $c > 0$. Since $\tau_0(E) \geq 0$, Chebychev's inequality leads to (4.8) as

$$\begin{aligned} &\mathbf{P}_y \left[\tau^\beta_k(E) \leq (1 - \delta) T_{r,R}(E) k \right] \leq \exp(\theta(1 - \delta) T_{r,R}(E) k) \mathbf{E}_y e^{-\theta \tau^\beta_k(E)} \\ &\leq \exp(\theta(1 - \delta) T_{r,R}(E) k) \left[\max_{y \notin E(R)} \mathbf{E}_y e^{-\theta \tau_0(E)} \right]^k \\ &\leq \exp(\theta(1 - \delta) T_{r,R}(E) k) \exp(\rho\theta^2 k - \theta(1 - \delta/2) T_{r,R}(E) k) \end{aligned}$$

Taking

$$\theta = \frac{\delta T_{r,R}(E)}{c_1 \rho}$$

we get that

$$\begin{aligned}&\mathbf{P}_y\left[\tau_k^\beta(E) \leq (1-\delta)T_{r,R}(E)k\right] \leq \exp(\rho\theta^2 k - \theta T_{r,R}(E)k\delta/2)\\ &\leq \exp(\rho\delta^2 T_{r,R}^2(E)k/(c_1^2\rho^2) - \delta^2 T_{r,R}^2(E)k/(2c_1\rho)) \leq \exp(-c\delta^2 k),\end{aligned}$$

provided we take c_1 sufficiently large.

To prove (4.9), we need to bound

$$\begin{aligned}&\mathbf{P}_y\left[\tau_k^\beta(E) \geq (1+\delta)T_{r,R}(E)k\right]\\ &\leq \exp(-\theta(1+\delta)T_{r,R}(E)k)\left(e^{\theta T_\beta^U}\max_y \mathbf{E}_y e^{\theta\tau_0(E)} \max_{y\in E(r)} \mathbf{E}_y e^{\theta[\sigma_0(E)-\tau_0(E)]}\right)^k\end{aligned}$$

We note that

$$\begin{aligned}&\max_y \mathbf{E}_y e^{\theta\tau_0(E)} \leq (1+o(1))\max_{y\notin E(R)} \mathbf{E}_y e^{\theta\tau_0(E)}\\ &\leq \exp(\theta(1+\delta/2)T_{r,R}(E) + \rho\theta^2 + o(1)).\end{aligned}$$

Take

$$\theta = \frac{\delta T_{r,R}(E)}{c_1\rho},$$

with c_1 to be fixed shortly. Since $\max_{y\in E(r)} \mathbf{E}_y[\sigma_0(E)-\tau_0(E)] = o(T_{r,R}(E))$ as $n\to\infty$, Kac's formula yields

$$\max_{y\in E(r)} \mathbf{E}_y e^{\theta[\sigma_0(E)-\tau_0(E)]} = 1+o(1) \text{ as } n\to\infty.$$

Since $T_\beta^U = o(T_{r,R}(E))$ as $n\to\infty$ as well, we have

$$\begin{aligned}&\mathbf{P}_y[\tau_k^\beta(E) \geq (1+\delta)T_{r,R}(E)k]\\ &\leq \exp(-\theta(1+\delta)T_{r,R}(E)k + \theta(1+\delta/2)T_{r,R}(E)k + \rho\theta^2 k + o(1)k)\\ &\leq \exp(-\theta\delta T_{r,R}(E)k/2 + \rho\theta^2 k + o(1)k).\end{aligned}$$

Taking $c_1 > 0$ large enough gives the result. □

4.3. *Hitting and Covering.*

Lemma 4.6 (Hitting Time Estimate). *For every $\delta > 0$ there exists r_0 such that for each $r \geq r_0$ there is an $R_0 > r$ so that if $R \geq R_0$ the following*

holds. If $E_n = \{x_{n1}, \ldots, x_{n\ell}\} \subseteq V_n$ *with* $d(x_{ni}, x_{nj}) \geq 2R$ *for* $i \neq j$ *and* $y_n \in V_n$ *is such that* $d(x_{ni}, y_n) \geq 2R$ *for all* n*, then*

$$1 - \delta \leq \liminf_{n\to\infty} \frac{\mathbf{E}_{y_n}\tau(x_{ni})}{T_{r,R}(E_n)/\overline{p}_{r,R}(x_{ni};E)} \tag{4.11}$$

$$\leq \limsup_{n\to\infty} \frac{\mathbf{E}_{y_n}\tau(x_{ni})}{T_{r,R}(E_n)/\overline{p}_{r,R}(x_{ni};E)} \leq 1 + \delta. \tag{4.12}$$

Proof. We will omit the indices n and i and just write x for x_{ni}, y for y_n, and E for E_n. Fix r sufficiently large so that $\rho(r) < \delta^2/100$. Let $N(x;E) = \min\{k \geq 1 : \mathbf{1}_{S_k^{\alpha,\beta}(x)} = 1\}$ and let

$$\widetilde{\tau}(x) = \min\{t \geq 0 : X(t) = x \text{ and } t \in I\}$$

where

$$I_k = [\tau_k^\beta(x), \sigma_k^\beta(x) + T_\alpha^U] \text{ and } I = \cup_k I_k.$$

Then

$$\tau^\beta_{N(x;E)}(E) \leq \widetilde{\tau}(x) \leq \tau^\beta_{N(x;E)+1}(E).$$

Let

$$A(M;\delta) = \bigcap_{j\geq M} B(j;\delta) \equiv \bigcap_{j\geq M} \left\{(1-\delta)T_{r,R}(E)j \leq \tau_j^\beta(E) \leq (1+\delta)T_{r,R}(E)j\right\}.$$

With $\|\widetilde{\tau}(x)\| = \max_z \mathbf{E}_z\widetilde{\tau}(x)$, note that

$$\begin{aligned}
\mathbf{E}_y\widetilde{\tau}(x)\mathbf{1}_{A^c(M;\delta)} &\leq \sum_{j\geq M} \mathbf{E}_y\widetilde{\tau}(x)\mathbf{1}_{B^c(j;\delta)} \\
&\leq \sum_{j\geq M}\left[\mathbf{E}_y\tau_j^\beta(E)\mathbf{1}_{B^c(j;\delta)} + \|\widetilde{\tau}(x)\|\mathbf{P}[B^c(j;\delta)]\right] \\
&\leq 2C\sum_{j\geq M}\left[jT_{r,R}(E) + \|\widetilde{\tau}(x)\|\right]e^{-C\delta^2 j} \\
&\leq C_1\|\widetilde{\tau}(x)\|\sum_{j\geq M}(1+j)e^{-C\delta^2 j} \leq C_2\|\widetilde{\tau}(x)\|\frac{e^{-C\delta^2 M}}{\delta^4}.
\end{aligned} \tag{4.13}$$

In the third step we used that

$$\begin{aligned}
\mathbf{E}_y\tau_j^\beta(E)\mathbf{1}_{B^c(j;\delta)} &\leq (\mathbf{E}_y[\tau_j^\beta(E)]^2)^{1/2}(\mathbf{P}[B^c(j;\delta)])^{1/2} \\
&\leq \frac{2T_{r,R}(E)}{\lambda} j\left(\mathbf{E}_y\exp(\lambda\tau_j^\beta(E)/(jT_{r,R}(E)))\right)^{1/2} Ce^{-C\delta^2 j},
\end{aligned}$$

where $\lambda \in (0, \lambda_0)$, λ_0 as in the proof of Lemma 4.5. Uniform local transience implies

$$\left|\mathbf{E}_y \widetilde{\tau}(x) - \|\widetilde{\tau}(x)\|\right| \leq \delta \mathbf{E}_y \widetilde{\tau}(x)$$

when R is large enough. Consequently, there exists $M > 0$ large enough depending only on δ so that

$$\mathbf{E}_y \widetilde{\tau}(x) \mathbf{1}_{A(M;\delta)} \leq \mathbf{E}_y \widetilde{\tau}(x) \leq (1+\delta) \mathbf{E}_y \widetilde{\tau}(x) \mathbf{1}_{A(M;\delta)}.$$

Now,

$$\begin{aligned} &\mathbf{E}_y \tau^{\beta}_{N(x;E)+1}(E) \mathbf{1}_{A(M;\delta)} = \mathbf{E}_y \left[N(x;E) \left(\frac{\tau^{\beta}_{N(x;E)+1}(E)}{N(x;E)} \right) \mathbf{1}_{A(M;\delta)} \right] \\ \leq &(1+\delta) T_{r,R}(x) \mathbf{E}_y N(x;E) + \mathbf{E}_y \tau^{\beta}_M(E) \leq (1+\delta) T_{r,R}(E) \mathbf{E}_y N(x) + CMT_{r,R}(E). \end{aligned}$$

and, similarly,

$$\mathbf{E}_y \tau_{N(x;E)}(E) \mathbf{1}_{A(M;\delta)} \geq (1-\delta) T_{r,R}(E) \mathbf{E}_y N(x;E).$$

Therefore

$$\mathbf{E}_y \widetilde{\tau}(x) \leq (1+2\delta) T_{r,R}(E) \mathbf{E}_y N(x;E) + CMT_{r,R}(E).$$

By Lemma 4.2,

$$\begin{aligned} &\overline{p}_{r,R}(x;E) \left[\exp(-(1+\delta)\overline{p}_{r,R}(x;E) j) - C \exp\left(-\frac{C\delta^2 \overline{p}_{r,R}(x;E)}{\rho(r)} j \right) \right] \\ \leq &\mathbf{E}_y p^{\alpha,\beta}_{j+1}(x;E) \exp\left(-[1 + O(\rho(r))] \sum_{i=1}^{j} p^{\alpha,\beta}_i(x;E) \right) \\ \leq &\overline{p}_{r,R}(x;E) \left[\exp(-(1-\delta)\overline{p}_{r,R}(x;E) j) + C \exp\left(-\frac{C\delta^2 \overline{p}_{r,R}(x;E)}{\rho(r)} j \right) \right]. \end{aligned}$$

Taking r sufficiently large gives

$$\begin{aligned} &\mathbf{E}_y N(x;E) = \sum_{j=1}^{\infty} j \mathbf{P}[N(x;E) = j] \\ \leq &CM^2 \rho(r) + \sum_{j=M+1}^{\infty} j(1+o(1)) \left(\overline{p}_{r,R}(x;E) \exp(-(1-\delta)\overline{p}_{r,R}(x;E) j) \right) \\ \leq &2CM^2 \rho(r) + \frac{1+\delta}{\overline{p}_{r,R}(x;E)}. \end{aligned}$$

Similarly,

$$\mathbf{E}_y N(x;E) \geq \frac{1-\delta}{\overline{p}_{r,R}(x;E)}.$$

Increasing r if necessary so that $M^2\rho(r) \leq \delta$ yields

$$\frac{1-2\delta}{\overline{p}_{r,R}(x;E)} \leq \mathbf{E}_y N(x;E) \leq \frac{1+2\delta}{\overline{p}_{r,R}(x;E)}.$$

This proves that $\mathbf{E}_y\widetilde{\tau}(x) = (1+o(1))\frac{T_{r,R}(E)}{\overline{p}_{r,R}(x;E)}$ as $n \to \infty$. By mixing considerations, the probability of the event F_k that X hits $E(r)$ in $J_k = [\sigma_k^\beta(E) + T_\alpha^U, \sigma_k^\beta(E) + T_\beta^U]$ is $O(T_\beta^U |E| \overline{\Delta}^r(G)/|V|)$. With $F = \cup_{k=1}^{N(x;E)+1} F_k$, we have

$$\mathbf{E}_y \widetilde{\tau}(x)\mathbf{1}_{F^c} \leq \mathbf{E}_y\tau(x) \leq \mathbf{E}_y\widetilde{\tau}(x)$$

and, analogous to the proof of (4.13),

$$\mathbf{E}_y\widetilde{\tau}(x)\mathbf{1}_{F^c} = \left[1 + O\left(\frac{T_\beta^U |E| \overline{\Delta}^r(G)}{|V|\overline{p}_{r,R}(x;E)}\right)^{1/2}\right]\mathbf{E}_y\widetilde{\tau}(x).$$

The lemma now follows since $\overline{p}_{r,R}(x;E) \geq C\underline{\Delta}^{-r}(G)$. □

We will now specialize to the case $E = \{x\}$; for simplicity of notation we will omit E. Let

$$O_{r,R}(x) = \frac{\overline{a}_{r,R}(x)}{T_{r,R}(x)}.$$

Lemma 4.7. *For every $\delta > 0$ there exists r_0 so if $r \geq r_0$ there is $R_0 > r$ such that $R \geq R_0$ implies*

$$(1-\delta)\pi(x) \leq O_{r,R}(x) \leq (1+\delta)\pi(x)$$

for all n large enough.

Proof. Let $N(x,T) = \min\{k : \tau_k^\beta(x) \geq T\}$, J_k as in the previous lemma, $J = \cup_k J_k$, and $\mathcal{G}(x) = \sigma(X(\tau_j^\beta(x)) : j \geq 1)$. Then

$$\sum_{j=1}^{N(x,T)} a_j^{\alpha,\beta}(x) \leq \mathbf{E}\left[\sum_{t=1}^{T} \mathbf{1}_{\{X(t)=x\}}\mathbf{1}_{t\notin J}\middle|\mathcal{G}(x)\right] \leq \sum_{j=1}^{N(x,T)+1} a_j^{\alpha,\beta}(x).$$

Lemmas 4.2 and 4.5 give that

$$(1-\delta)T_{r,R}(x) \leq \frac{N(x,T)}{T} \leq (1+\delta)T_{r,R}(x) \text{ and}$$

$$(1-\delta)\overline{a}_{r,R}(x) \leq \frac{\sum_{j=1}^k a_j^{\alpha,\beta}(x)}{k}(1+\delta)\overline{a}_{r,R}(x),$$

with high probability as $T \to \infty$, for all $r, R, k, n, \beta - \alpha$ large enough. Consequently, using that $(a_j^{\alpha,\beta}(x) : j \geq 1)$ is uniformly bounded, it is not hard to see that

$$(1-\delta)\frac{\overline{a}_{r,R}(x)}{T_{r,R}(x)} \leq \frac{1}{T}\sum_{j=1}^{N(x,T)} a_j^{\alpha,\beta}(x) \leq (1+\delta)\frac{\overline{a}_{r,R}(x)}{T_{r,R}(x)},$$

with high probability as $T \to \infty$, for all $r, R, n, \beta - \alpha$ large enough. The middle term converges to $\pi(x)$ as $T \to \infty$ since

$$\lim_{T\to\infty} \frac{1}{T}\mathbf{E}\sum_{t=1}^{T} \mathbf{1}_{\{X(t)\in E(r)\}}\mathbf{1}_{t\in J} = 0.$$

$\square$

Uniform local transience implies that there exists constants $c, C > 0$ so that $c\overline{a}_{r,R}(x) \leq \overline{p}_{r,R}(x) \leq C\overline{a}_{r,R}(x)$; combining this with the previous lemma yields

$$\frac{c\deg(x)}{|V|} \leq \frac{\overline{p}_{r,R}(x)}{T_{r,R}(x)} \leq \frac{C\deg(x)}{|V|}.$$

Let $\epsilon > 0$ and let

$$H_{n,k}^{\epsilon} = \left\{ x \in V_n : \frac{\underline{\Delta}(G_n)k\epsilon}{|V_n|} < \frac{\overline{p}_{r,R}(x)}{T_{r,R}(x)} \leq \frac{\underline{\Delta}(G_n)(k+1)\epsilon}{|V_n|} \right\}$$

be a partition of V_n into at most $\Delta_0\epsilon^{-1}$ subsets, where Δ_0 is the constant from Assumption 1.1. By passing to a subsequence, we may assume without loss of generality that

$$d_k^{\epsilon} = \lim_{n\to\infty} d_{n,k}^{\epsilon} \equiv \lim_{n\to\infty} \frac{\log|H_{n,k}^{\epsilon}|}{\log|V_n|}$$

exists for every k. Note that $d_k^{\epsilon} \in [0,1]$ for those k so that $|H_{n,k}^{\epsilon}| \neq 0$ for all n large enough and, since the partition is finite, necessarily there exists k so that $d_k^{\epsilon} = 1$. In particular, there exists k so that $d_k^{\epsilon} \neq 0$. Let

$$C_{n,k}^{\epsilon} = \frac{|V_n|}{\underline{\Delta}(G_n)k\epsilon} d_k^{\epsilon}\log|V_n| \text{ and } C_n^{\epsilon} = \max_k C_{n,k}^{\epsilon}. \tag{4.14}$$

LEMMA 4.8 (Cover Time Estimate). *For each $\delta > 0$ there exists r_0, ϵ_0 so if $r \geq r_0$ there is $R_0 > r$ such that $R \geq R_0$ and $\epsilon \in (0,\epsilon_0)$ implies*

$$1-\delta \leq \liminf_{n\to\infty} \frac{T_{\mathrm{cov}}(H_{n,k}^{\epsilon})}{C_{n,k}^{\epsilon}} \leq \limsup_{n\to\infty} \frac{T_{\mathrm{cov}}(H_{n,k}^{\epsilon})}{C_{n,k}^{\epsilon}} \leq 1+\delta. \tag{4.15}$$

for all k with $d_k^\epsilon > 0$. Furthermore,

$$1 - \delta \leq \liminf_{n\to\infty} \frac{T_{\mathrm{cov}}(G_n)}{C_n^\epsilon} \leq \limsup_{n\to\infty} \frac{T_{\mathrm{cov}}(G_n)}{C_n^\epsilon} \leq 1 + \delta \tag{4.16}$$

Proof. Suppose k is such that $d_k^\epsilon > 0$. Then $|H_{n,k}^\epsilon| \to \infty$ as $n \to \infty$. Let $r, R, n > 0$ be sufficiently large so that Lem ma 4.6 applies with our choice of δ. Since $\log|B(x,r)| = o(\log|V_n|)$, it follows that for all n large enough there exists an R-net $E_{n,k}^\epsilon$ of $H_{n,k}^\epsilon$ such that

$$\log|E_{n,k}^\epsilon| = \log|H_{n,k}^\epsilon| + o(1) \text{ as } n \to \infty.$$

The upper and lower bounds from the Matthews method ([22]; see also Theorem 11.2 and Proposition 11.4 of [21]) combined with the definition of $C_{n,k}^\epsilon$ imply (4.15). Theorem 2 of [2] implies that

$$\lim_{n\to\infty} \frac{\tau_{\mathrm{cov}}(H_{n,k}^\epsilon)}{\mathbf{E}\tau_{\mathrm{cov}}(H_{n,k}^\epsilon)} = 1.$$

As $\tau_{\mathrm{cov}}(G_n) = \max_k \tau_{\mathrm{cov}}(H_{n,k}^\epsilon)$ and the maximum is over a finite set, it follows that $\tau_{\mathrm{cov}}(G_n) = (1 + o(1)) \max_k T_{\mathrm{cov}}(H_{n,k}^\epsilon)$. Taking expectations of both sides gives (4.16). □

5. Correlation Decay. Note that vertex transitivity implies $\overline{p}_{r,R}(\cdot)$ and $T_{r,R}(\cdot)$ do not depend on their arguments.

Proof of Theorem 1.6. First, assume that we are in the case of bounded maximal degree. Let E be as in the previous section and let $\delta > 0$ be arbitrary. Fix r so that $\rho(r) \leq \delta^3/100C\ell$ and $\overline{p}_{r,R} \leq \delta^3$ where $\ell = |E|$. Let $R_0 > r$ and $\beta - \alpha$ be sufficiently large so that Lemmas 4.2 and 4.5 apply with our choice of δ, r. Finally, let $N(x_i) = \min\{k : S_k^{\alpha,\beta}(x_i) \text{ occurs}\}$ and $\mathcal{G}(E) = \sigma(p_j^{\alpha,\beta}(x;E) : x \in E, j \geq 1)$. Since $d(x_i, x_j) \geq 2R$, the probability that X neither hits x nor x' in the interval $[\tau_j^\beta(x;E), \sigma_j^\beta(x;E) + T_\alpha^U]$ is

$$1 - [1 + O(\rho(R))][p_j^{\alpha,\beta}(x;E) + p_j^{\alpha,\beta}(x';E)]. \tag{5.1}$$

This holds more generally for any subset of E, hence

$$\begin{aligned}
&\mathbf{E}[\mathbf{P}[N(x_1) > k_1, \ldots, N(x_\ell) > k_\ell | \mathcal{G}(E)]] \\
=&\mathbf{E}\prod_{i=1}^{\ell} \exp\left(-[1 + O(\rho(r))]\sum_{j=1}^{k_i} p_j^{\alpha,\beta}(x_i;E)\right) \\
=&\exp\left(-[1 + O(\delta)]\sum_{i=1}^{\ell} \overline{p}_{r,R}(x_i;E)k_i\right) + \sum_{i=1}^{\ell} O(\exp(-\overline{p}_{r,R}(x_i;E)\ell k_i/\delta)),
\end{aligned}$$

where the last equality followed from our choice of r and Lemma 4.2. Combining this with Lemma 4.5 and that the probability X hits $E(r)$ in J_k is at most $O(T_\beta^U |E| \overline{\Delta}^r(G)/|V|) = o(\overline{p}_{r,R}(x;E))$ for any $x \in E$, we have

$$\begin{aligned}&\mathbf{P}[\tau(x_1) \geq kT_{r,R}(E)/\overline{p}_{r,R}(x_1;E), \ldots, \tau(x_\ell) \geq kT_{r,R}(E)/\overline{p}_{r,R}(x_n;E)]\\ =&(1+o(1))\exp\left(-[1+O(\delta)]\ell k\right) + O(\exp(-C\delta^2 k/\rho(r)))\\ =&(1+o(1))\exp\left(-[1+O(\delta)]\ell k\right)\end{aligned}$$

By vertex transitivity,

$$T_{\text{hit}}(G) = (1+o(1))\frac{T_{r,R}(x_i)}{\overline{p}_{r,R}(x_i)} = (1+o(1))\frac{T_{r,R}(E)}{\overline{p}_{r,R}(x_i;E)}.$$

Using that the cover time is asymptotically $T_{\text{hit}}(G)\log|V|$ gives the result.

This proof works also for unbounded degree, but is not quite sufficient for the statement of our theorem since we would like to allow for points in E to be adjacent. There are two parts that break down. First, in Section 4 we proved the concentration of $p_j^{\alpha,\beta}(x;E)$ when $x \in E$ and we also assumed that $x, y \in E$ implies $d(x,y) \geq 2R$. To allow for x, y adjacent, we define

$$p_j^{\alpha,\beta}(y;E) = \mathbf{P}[S_j^{\alpha,\beta}(y;E) | X(\tau_j^\beta(E)), X(\tau_{j+1}^\beta(E))]$$

for $y \in E(r/2)$. It is not difficult to see that for such y, $p_j^{\alpha,\beta}(y;E)$ exhibits nearly the same concentration behavior as for $y \in E$. Second, the estimate (5.1) is no longer good enough since $\rho(1)$ does not decay in n. However, it is not difficult to see that the same probability satisfies the estimate

$$1 - [1 + O(\overline{\Delta}^{-1}(G))][p_j^{\alpha,\beta}(x;E) + p_j^{\alpha,\beta}(x';E)], \tag{5.2}$$

which suffices since $\overline{\Delta}^{-1}(G_n) \to 0$ as $n \to \infty$. The rest of the proof is the same. □

Vertex transitivity was used only to get that $T_{r,R}(x;E)/\overline{p}_{r,R}(x;E) = (1+o(1))T_{\text{hit}}(G)$. The same proof works more generally, but leads to more complicated formulae. However, it is not difficult to see that the upper bound takes a very similar form:

Lemma 5.1. *If (x_n^i) for $1 \leq i \leq \ell$ is a family of sequences with $x_n^i \in H_{n,k(i)}^\epsilon$ and $|x_n^i - x_n^j| \geq r$ for every n and $i \neq j$,*

$$\mathbf{P}[x_n^i \in \mathcal{L}(\alpha;G_n) \text{ for all } i] \leq (1+\delta_{r,\ell})|V_n|^{-\ell d_k^\epsilon \alpha + \delta_{r,\ell}} \tag{5.3}$$

where $\delta_{r,\ell} \to 0$ as $r \to \infty$ while ℓ is fixed. If $\overline{\Delta}(G_n) \to \infty$ then we take $r = 1$ and $\delta_{1,\ell} = o(1)$ as $n \to \infty$.

6. Total Variation Bounds.

6.1. *Lower Bound.* We will now prove the lower bound for Theorems 1.3 and 1.5. This is actually just a slight extension of Theorem 4.1 of [23], but we include it for the reader's convenience. Recall from the introduction that $\mu(\cdot;\alpha,G)$ is the probability measure on $\mathcal{X}(G)=\{f\colon V\to\{0,1\}\}$ given by first sampling $\mathcal{R}(\alpha;G)$ then setting

$$f(x)=\begin{cases}\xi(x) \text{ if } x\in\mathcal{R}(\alpha;G),\\ 0 \text{ otherwise,}\end{cases}$$

where $(\xi(x):x\in V)$ is a collection of iid variables such that $\mathbf{P}[\xi(x)=0]=\mathbf{P}[\xi(x)=1]=\frac{1}{2}$ and $\nu(\cdot;G)$ is the uniform measure on $\mathcal{X}(G)$.

Lemma 6.1 (Lower Bound). *For every $\delta>0$,*

$$\lim_{n\to\infty}\|\mu(\cdot;\tfrac{1}{2}-\delta,G_n)-\nu(\cdot;G_n)\|_{TV}=1.$$

Proof. For $A\subseteq V$ and $m>0$, let $\tau_{\mathrm{cov}}(A;m)$ be the first time all but m of the vertices of A have been visited by X. For each k such that $d_k^\epsilon>0$ we will show that

$$\lim_{n\to\infty}\mathbf{P}[\tau_{\mathrm{cov}}(H_{n,k}^\epsilon,|H_{n,k}^\epsilon|^\alpha)<(1-\alpha-\delta)C_{n,k}^\epsilon]=0 \tag{6.1}$$

for each $\delta>0$ and $\epsilon\in(0,\epsilon_0(\delta))$. If not, then for some such k,δ,α we have

$$\limsup_{n\to\infty}\mathbf{P}[A_{n,k}(\alpha,\delta)]>0$$

where

$$A_{n,k}(\alpha,\delta)=\{\tau_{\mathrm{cov}}(H_{n,k}^\epsilon,|H_{n,k}^\epsilon|^\alpha)<(1-\alpha-\delta)C_{n,k}^\epsilon\}.$$

It follows from the Matthews method upper bound ([22]; see also Theorem 11.2 of [21]) that

$$\begin{aligned}&\mathbf{E}[\tau_{\mathrm{cov}}(H_{n,k}^\epsilon)-\tau_{\mathrm{cov}}(H_{n,k}^\epsilon,|H_{n,k}^\epsilon|^\alpha)|A_{n,k}(\alpha,\delta)]\\ \le&\alpha(1+O(\epsilon))C_{n,k}^\epsilon\le\alpha(1+\delta/4)C_{n,k}^\epsilon,\end{aligned}$$

where we take ϵ so small that the $O(\epsilon)$ term is at most $\delta/4$. Markov's inequality now implies

$$\mathbf{P}[\tau_{\mathrm{cov}}(H_{n,k}^\epsilon)<(1-\delta/2)C_{n,k}^\epsilon|A_{n,k}(\alpha,\delta)]>0.$$

This is a contradiction as Theorem 2 of [2] implies $\tau_{\mathrm{cov}}(H_{n,k}^\epsilon)/C_{n,k}^\epsilon\to1$ in probability.

For each n let $k_0(n)$ be an index that achieves the maximum in $\max_k C^\epsilon_{n,k}$. Now, (6.1) implies that whp at time $\frac{1}{2}(1-3\delta)T_{\rm cov}(G_n) = \frac{1}{2}(1-3\delta+O(\epsilon))C^\epsilon_{n,k_0(n)}$ the size of the subset of $H^\epsilon_{n,k_0(n)}$ not visited by X is at least $|H^\epsilon_{n,k_0(n)}|^{(1+2\delta+O(\epsilon))/2}$ but less than $|H^\epsilon_{n,k_0(n)}|^{(1+4\delta+O(\epsilon))/2}$. Thus the number of zeros in a marking of $H^\epsilon_{n,k_0(n)}$ sampled from $\mu(\cdot;\frac{1}{2}(1-3\delta),G_n)$ is whp at least

$$\tfrac{1}{2}|H^\epsilon_{n,k_0(n)}| + (1+o(1))|H^\epsilon_{n,k_0(n)}|^{(1+2\delta+O(\epsilon))/2} \text{ as } n\to\infty.$$

This proves the lemma since the probability of having deviations of this magnitude from the mean tends to zero in a uniform marking. □

6.2. *Concentration of q_j.* Let $q_j(x) = \mathbf{P}[S_j(x)|X(\tau_j(x)),X(\sigma_j(x))]$. Note that $q_j(x)$ is the not the same as $p_j^{\alpha,\beta}(x)$ from Section 4. Indeed, the excursions on which we condition are different since we do not allow the random walk to run for a multiple for $T^U_{\rm mix}(G)$ after exiting $\partial B(x,R)$ and we condition on the entrance and exit points of the current excursion rather than the entrance points of the current and successive excursion. While both of these changes may seem cosmetic, they affect the concentration behavior, since while $p_j^{\alpha,\beta}(x)$ satisfies (4.2), in locally tree-like graphs it can be that $q_j(x)=1$ with positive probability.

Suppose that (G_n) satisfies Assumption 1.2 part (1). Let $\epsilon>0$ be arbitrary, R_n^γ be as in Assumption 1.2, $\gamma>0$ to be determined later, and let E be a set of points in V_n such that if x,y are distinct in E then $d(x,y)\geq 4R_n^\gamma$. Fix $R>r>0$ and let $\tau_{k+1}(E) = \min\{t\geq \sigma_k(x) : X(t)\in\partial E(r)\}$. Fix $\beta>0$ and define indices $i(j,x)$ inductively as follows. Set

$$i(1,x) = \min\{k\geq 1 : \tau_{k+1}(E)-\sigma_k(x)\geq T^U_\beta\}$$

and, for each $j\geq 1$, let

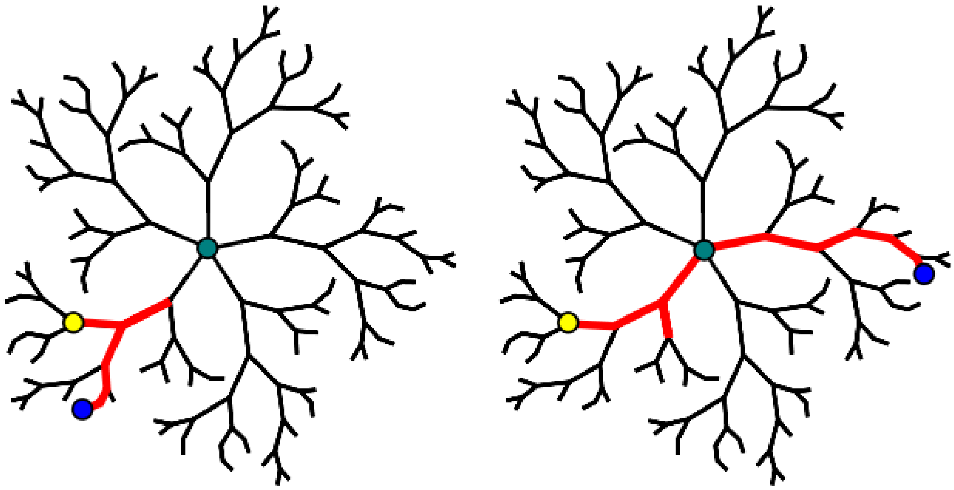
$$i(j+1,x) = \min\{k\geq i(j,x)+1 : \tau_{k+1}(E)-\sigma_k(x)\geq T^U_\beta\}.$$

When x is clear from the context we will write $i(j)$ for $i(j;x)$.

LEMMA 6.2. *For each $\delta>0$ and $r>0$ there exists $R_0>r$ such that for $R>R_0$ fixed there exists iid random variables $(I(j,x) : x\in E, j\geq 1)$ which stochastically dominate from above $(i(j,x) : x\in E, j\geq 1)$ and satisfy*

$$\mathbf{P}[I((1-\delta)j,x)\geq j] \leq C\exp(-C\delta^2 j)$$

for all n large enough. Let $\mathcal{G}(j,x) = \sigma(\{q_{i(k)}(x) : k\neq j\}\cup\{q_{i(k)}(y) : y\in E\setminus\{x\}\})$. There exists iid random variables $(Q_j(x) : j\geq 1)$ taking values



(a) Entrance and exit points of an excursion from $B(x,4)$ to $B(x,6)$, respectively, conditional on which random walk has a low probability of hitting x.

(b) Entrance and exit points of an excursion from $B(x,4)$ to $B(x,6)$, respectively, conditional on which random walk is forced to hit x.

FIG 4. *The concentration behavior of the* $q_j(x)$ *is very different from the* $p_j^{\alpha,\beta}(x)$ *since it is not in general true that* $q_j(x) \leq C\rho(r)$ *while it is true that* $p_j^{\alpha,\beta}(x) \leq C\rho(r)$*. For example, in a graph which is locally tree like as depicted above, it can be that* $q_j(x) = 1$ *for some combinations of entrance and exit points.*

in $[0, 2\rho(r)]$ *such that*

$$1 - O(e^{-c\beta}) \leq \frac{\mathbf{E}[q_{i(j)}(x)|\mathcal{G}(j,x)]}{Q_j(x)} \leq 1 + O(e^{-c\beta})$$

and

$$1 - O(e^{-c\beta}) \leq \frac{\overline{p}_{r,R}(x)}{\mathbf{E}Q_j(x)} \leq 1 + O(e^{-c\beta})$$

for all n *large enough. Furthermore, the families* $\{(Q_j(x) : j \geq 1) : x \in E\}$ *are independent.*

PROOF. Define stopping times

$$\sigma_{k0}(E) = \min\{t \geq \sigma_k(x) : d(X(t), E) \geq 2R_n^\gamma\},$$
$$\tau_{k1}(E) = \min\{t \geq \sigma_{k0}(x) : d(X(t), E) \leq R_n^\gamma\}$$

For $j \geq 1$, inductively set

$$\sigma_{kj}(E) = \min\{t \geq \tau_{kj}(E) : d(X(t), E) \geq 2R_n^\gamma\},$$
$$\tau_{k(j+1)}(E) = \min\{t \geq \sigma_{kj}(E) : d(X(t), E) \leq R_n^\gamma\}.$$

Note that $\sigma_{kj}(E) - \tau_{kj}(E) \geq R_n^\gamma$. Thus, for $j_\beta = T_\beta^U / R_n^\gamma$ we have that $\tau_{kj_\beta}(E) \geq \sigma_k(x) + T_\beta^U$. Let $F_k(x) = \{X(t) \in E(r) \text{ for } t \in [\sigma_k(x), \sigma_k(x) + T_\beta^U]\}$. Let x_{kj} be the element in E such that $d(X(\tau_{kj}(E)), x_{kj}) \leq R_n^\gamma$. Observe

$$\begin{aligned}&\mathbf{P}_{X(\tau_{kj}(E))}[X(t) \in E(r) \text{ for } t \in [\tau_{kj}(E), \sigma_{kj}(E)] | x_{kj}]\\ \leq & C \max_{d(y,x_{kj}) = R_n^\gamma} g(y, B(x_{kj}, r); G_n).\end{aligned}$$

Uniform local transience also yields

$$\mathbf{P}_{X(\sigma_k(x))}[X(t) \in E(r) \text{ for } t \in [\sigma_k(x), \tau_{k0}(E)]] \leq C\rho(R, r) \leq \delta/2,$$

provided $R > r$ is large enough. A union bound thus gives

$$\begin{aligned}\mathbf{P}_{X(\tau_k(x))}[F_k(x)] &\leq \max_z \max_{d(y,z) = R_n^\gamma} g(y, B(z, r); G_n) \frac{T_\beta^U}{R_n^\gamma} + \delta/2\\ &\leq \delta/2 + o(1) \leq \delta, \end{aligned} \tag{6.2}$$

as $n \to \infty$ by part (1) of Assumption 1.2. Note that if $x_1, \ldots, x_\ell \in E$ and $j(1), \ldots, j(k)$ are such that $\tau_{j(k)}(x_k) \leq \tau_{j(k+1)}(x_{k+1})$ then we have

$$\begin{aligned}&\mathbf{P}[F_{j(1)}(x_1), \ldots, F_{j(\ell)}(x_\ell)] = \mathbf{E}[\mathbf{P}_{X(\tau_{j(\ell)})(x_\ell)}[F_{j(\ell)}(x_\ell)] \mathbf{1}_{F_{j(1)}(x_1)} \cdots \mathbf{1}_{F_{j(\ell-1)}(x_{\ell-1})}]\\ \leq & \delta \mathbf{P}[F_{j(1)}(x_1), \ldots, F_{j(\ell-1)}(x_{\ell-1})] \leq \cdots \leq \delta^\ell.\end{aligned}$$

This can of course be repeated with any subset of the above events which implies the stochastic domination claim. It easily now follows from Cramer's theorem that

$$\mathbf{P}\left[I((1-\delta)k, x) \geq k\right] \leq 2\exp(-C\delta^2 k).$$

For the second part of the lemma, we just need to get a bound on $\mu_x(z)/\pi(z)$ where μ_x is the law of random walk started at x conditioned not to get within distance r of E by, say, time $T_{\beta/2}^U$. This can be done in exactly the same way as in the proof of Lemma 4.3. Indeed, the term $|E|\rho(s, r)$ in the statement of that lemma comes from a bound on the probability that random walk at distance s from E hits E in time T_α^U. In the situation of this lemma, the role of s is replaced by R_n^γ and we can use the scheme developed above to estimate the error contributed by this term by $O(\delta)$ provided n is sufficiently large. □

LEMMA 6.3. *If (G_n) satisfies part (2) of Assumption 1.2 then for each $r, \delta > 0$ there exists $R_0 > r$ such that $R \geq R_0$ implies*

$$\begin{aligned} \mathbf{P}\left[\prod_{j=1}^{k}(1-q_j(x)) \geq (1-(1+\delta)\overline{p}_{r,R}(x))^{k(1+\delta)}\right] \\ \leq C\big[\exp\left(-C\delta^2\overline{p}_{r,R}(x)k/\rho(r)\right) + \exp(-C\delta^2 k)\big]. \end{aligned} \tag{6.3}$$

for all n large enough.

PROOF. The uniform Harnack inequality implies that $q_j(x) \leq 2C\rho(r)$ where $C = C(R/r)$ is the constant from the statement of part (2) of Assumption 1.2. Let $F_j = \{\tau_j(x) - \sigma_{j-1}(x) \leq T_\beta^U\}$. Arguing as in the previous lemma and invoking uniform local transience, there exists iid random variables $\widetilde{F}_j(x)$ with $\mathbf{P}[\widetilde{F}_j(x) = 1] = \delta = 1 - \mathbf{P}[\widetilde{F}_j(x) = 0]$ that stochastically dominate $(\mathbf{1}_{F_j(x)} : j)$ provided R is sufficiently large. We let $\iota(j)$ be the jth smallest index i such that $F_i(x)$ occurs. The lemma now follows from an argument similar to that of Lemma 4.2. Indeed, we can stochastically dominate $q_{\iota(j)}(x)$ from below by iid random variables L_j with $\mathbf{E}L_j \geq (1-\delta)\overline{p}_{r,R}(x)$ and $L_j \leq 10C\rho(r)$. By Cramer's theorem,

$$\mathbf{P}\left[\prod_{j=1}^{k}(1-L_j) \geq (1-(1+\delta)\overline{p}_{r,R}(x))^{k}\right] \leq C\exp\left(-C\delta^2\overline{p}_{r,R}(x)k/\rho(r)\right).$$

The lemma now follows since, again by Cramer's theorem,

$$\mathbf{P}[\iota((1-\delta)k) \geq k] \leq C\exp(-C\delta^2 k).$$

□

6.3. *Proof of Theorem 1.3.* Let $\delta > 0$ be arbitrary and assume that $R > r, n_0, \epsilon$ have been chosen so that for all $n \geq n_0$ we have

$$1 - \delta \leq \frac{T_{\text{cov}}(G_n)}{C_n^\epsilon} \leq 1 + \delta.$$

We may assume without loss of generality that $d_k^\epsilon > 0$ for all relevant k and, in particular, that $|H_{n,k}^\epsilon|^{-\delta} \to 0$ for every k. Indeed, Lemma 4.6 implies that $T_{\text{hit}}(G_n) = \Theta(|V_n|)$, consequently if $\log|H_{n,k}^\epsilon| \to 0$ as $n \to \infty$ then $T_{\text{cov}}(H_{n,k}^\epsilon)$ is negligible in comparison to $T_{\text{cov}}(G_n)$. If (G_n) satisfies Assumption 1.2 Part (1) we take R_n^γ as given there. Otherwise, we take $R_n^\gamma = \max\{R > 0 : \max_{x \in V_n} |B(x,R)| \leq |V_n|^\gamma\}$.

LEMMA 6.4. *Let $\mathcal{R}(t)$ denote the range of random walk at time t and $\mathcal{L}(t) = V \setminus \mathcal{R}(t)$. Letting*

$$M = \begin{cases} 20\Delta_0 \sup_n \overline{\Delta}^R(G_n)/(\delta\epsilon d^\epsilon) \text{ if } \sup_n \overline{\Delta}(G_n) < \infty, \\ 20\Delta_0/(\delta\epsilon d^\epsilon) \text{ otherwise} \end{cases}$$

and

$$\mathcal{T}_0 = \min\left\{T \geq 0 : \max_x |\mathcal{L}(t) \cap B(x, R_n^\gamma)| \leq M\right\},$$

we have that $\mathbf{P}[\mathcal{T}_0 > \frac{1+5\delta}{2} T_{\mathrm{cov}}(G_n)] = o(1)$ provided γ is sufficiently small, R is so large that $\delta_{R,m} \leq 1$, $d^\epsilon = \min\{d_k^\epsilon : d_k^\epsilon > 0\}$, and $m = 20/d^\epsilon$. Furthermore, letting

$$\mathcal{T}_1 = \min\left\{T \geq 0 : |\mathcal{L}(t) \cap H_{n,k}^\epsilon| \leq |H_{n,k}^\epsilon|^{1/2-\delta} \text{ for all } k\right\}$$

we have that $\mathbf{P}[\mathcal{T}_1 > \frac{1+5\delta}{2} T_{\mathrm{cov}}(G_n)] = o(1)$.

PROOF. First suppose that (G_n) has uniformly bounded maximal degree. Fix $R > r$ and let E be an R-net of $H_{n,k}^\epsilon$. Fix $x \in H_{n,k}^\epsilon$ and suppose that $x_1, \ldots, x_\ell \in B(x, R_n^\gamma) \cap H_{n,k}^\epsilon \cap E$ are distinct. Lemma 5.1 gives us

$$\mathbf{P}[x_1, \ldots, x_\ell \in \mathcal{L}((1+\delta)/2; G_n)] \leq (1 + \delta_{R,\ell})|V_n|^{-(1+\delta)\ell d_k^\epsilon/2 + \delta_{R,\ell}}.$$

Consequently, a union bound yields

$$\begin{aligned} &\mathbf{P}[|\mathcal{L}((1+\delta)/2; G_n) \cap B(x, R_n^\gamma) \cap E| \geq \ell] \\ \leq &(1 + \delta_{R,\ell})|B(x, R_n^\gamma)|^\ell |V_n|^{-(1+\delta)\ell d_k^\epsilon/2 + \delta_{R,\ell}} \\ \leq &(1 + \delta_{R,\ell})|V_n|^{(\gamma - (1+\delta)d_k^\epsilon/2)\ell + \delta_{R,\ell}}. \end{aligned}$$

Hence choosing $\gamma \leq d^\epsilon/4$ the above is $O(|V_n|^{-3})$. Since the number of disjoint R-nets necessary to cover $H_{n,k}^\epsilon$ is at most $\overline{\Delta}^R(G_n)$, the result now follows from a union bound. In the case of unbounded maximal degree we can skip the step of subdividing the $H_{n,k}^\epsilon$ into R-nets since in this case $\delta_{1,m} \to 0$, otherwise the proof is the same. The second claim is immediate from Markov's inequality and Lemma 5.1. □

Let $N(x, T)$ be the number of excursions from $\partial B(x, r)$ to $\partial B(x, R)$ that have occurred by time T.

Proof of Theorem 1.3, under Assumption 1.2 part (2). Let

$$\mathcal{T}_2 = \min\left\{ T \geq 0 : \max_{x \in H^\epsilon_{n,k}} \prod_{k=1}^{N(x,T)} (1 - q_j(x)) \leq |H^\epsilon_{n,k}|^{-1/2-\delta} \text{ for all } k \right\}.$$

and set

$$\mathcal{T} = \mathcal{T}_0 \vee \mathcal{T}_1 \vee \mathcal{T}_2 \vee \left(\frac{1+5\delta}{2}\right) T_{\mathrm{cov}}(G). \tag{6.4}$$

Let $k_0(n)$ be a sequence so that $\liminf_{n\to\infty} d^\epsilon_{k(n)} \geq \delta_0 > 0$. For $x \in H^\epsilon_{n,k_0(n)}$ we have

$$\left(\frac{1+3\delta}{2}\right) C^\epsilon_{n,k_0(n)} \geq \left(\frac{1+3\delta+O(\epsilon)}{2}\right) \frac{\delta_0 T_{r,R}(x) \log|V_n|}{4\rho(r)}$$

for all n large enough. Thus letting $M^\epsilon_{n,k_0(n)}(x) = (1+3\delta)/2 \cdot C^\epsilon_n(x)/T_{r,R}(x)$, we have

$$M^\epsilon_{n,k_0(n)}(x) \geq \left(\frac{1+3\delta+O(\epsilon)}{2}\right) \frac{\delta_0 \log|V_n|}{4\rho(r)}.$$

Now,

$$\begin{aligned} &\mathbf{P}[(1-\delta)T_{r,R}(x)M^\epsilon_{n,k_0(n)}(x) \leq \tau_{M^\epsilon_{n,k_0(n)}}(x) \leq (1+\delta)T_{r,R}(x)M^\epsilon_{n,k_0(n)}(x)] \\ \geq& 1 - C\exp\left(-\frac{C\delta_0\delta^2}{\rho(r)}\log|V_n|\right) \geq 1 - O(|V|^{-100}), \end{aligned}$$

provided we choose r large enough. Choosing $R > r$ sufficiently large, Lemma 6.3 gives us

$$\mathbf{P}\left[\prod_{j=1}^{M^\epsilon_{n,k_0(n)}(x)} (1-q_j(x)) \geq |H^\epsilon_{n,k_0(n)}|^{-1/2-\delta}\right] \leq O(|V|^{-100}).$$

Combining everything,

$$\mathbf{P}\left[\mathcal{T} \neq \left(\frac{1+5\delta}{2}\right) T_{\mathrm{cov}}(G)\right] = o(1) \text{ as } n \to \infty. \tag{6.5}$$

Let μ be the probability on $\mathcal{X}(G)$ given by first sampling $\mathcal{R} \subseteq G$ according to μ_0, the measure on subsets of V given by running X to time $(1+5\delta)/2 \cdot T_{\mathrm{cov}}(G)$, then sampling $f|\mathcal{R}$ by marking with iid fair coins and $f|V \setminus \mathcal{R} \equiv 0$. Define $\widetilde{\mu}$ similarly except by sampling $\mathcal{R} \subseteq G$ according to $\widetilde{\mu}_0$, the measure

given by running X up to time $\mathcal{T}$ rather than $(1+5\delta)/2 \cdot T_{\text{cov}}(G)$. As a consequence of (6.5),

$$\|\mu - \widetilde{\mu}\|_{TV} \leq \mathbf{P}\left[\mathcal{T} \neq \left(\frac{1+5\delta}{2}\right) T_{\text{cov}}(G)\right] = o(1) \text{ as } n \to \infty.$$

Suppose we have two independent random walks X, X' on G, each with stationary initial distribution, and let $\mathcal{T}, \mathcal{T}'$ be stopping times for each as in (6.4). Let $\mathcal{R}, \mathcal{R}'$ be their ranges at time $\mathcal{T}, \mathcal{T}'$, respectively, and $\mathcal{L} = V \setminus \mathcal{R}$, $\mathcal{L}' = V \setminus \mathcal{R}'$. Let $q_j'(x)$ be the quantity analogous to $q_j(x)$ for X' and $\mathcal{G} = \sigma(q_j'(x) : j \geq 1)$. The previous lemma implies that we can divide $\mathcal{L}$ into M disjoint sets $E_1, \ldots, E_M$ such that if $x, y \in E_\ell$ with $x \neq y$ then $d(x,y) \geq R_n^\gamma > R$. Consequently, letting $\mathcal{G}(E_\ell) = \otimes_{x \in E_\ell} \mathcal{G}(x)$ we have

$$\begin{aligned}\mathbf{E}[\exp(\zeta|\mathcal{L} \cap \mathcal{L}' \cap E_\ell|)|\mathcal{G}(E_\ell)] &\leq \prod_{x \in E_\ell}\left(1 + e^\zeta \left(\prod_{j=1}^{N(x,\mathcal{T}')} (1 - q_j'(x))\right)\right)\\ &\leq \exp\left(e^\zeta \sum_k |H_{n,k}^\epsilon|^{-\delta}\right).\end{aligned}$$

Since $E_1, \ldots, E_M$ cover $\mathcal{L}$, it follows from Hölder's inequality that

$$\begin{aligned}\mathbf{E}\exp(\zeta|\mathcal{L} \cap \mathcal{L}'|) &\leq \left[\exp(e^{\zeta M} \sum_k |H_{n,k}^\epsilon|^{-\delta})\right]^{1/M}\\ &\leq 1 + 2\frac{\exp(\zeta M)}{M} \sum_k |H_{n,k}^\epsilon|^{-\delta}.\end{aligned} \tag{6.6}$$

□

Proof of Theorem 1.3, under Assumption 1.2 Part (1). Let

$$\mathcal{T}_2 = \min\left\{T \geq 0 : \max_k \max_{x \in H_{n,k}^\epsilon} \frac{(1+2\delta)\log|H_{n,k}^\epsilon|}{2N(x,T)\overline{p}_{r,R}(x)} \leq 1\right\}$$

and

$$\mathcal{T} = \mathcal{T}_0 \vee \mathcal{T}_1 \vee \mathcal{T}_2 \vee \left(\frac{1+5\delta}{2}\right) T_{\text{cov}}(G). \tag{6.7}$$

It follows from Lemmas 4.5 and 4.8 and the definition of $H_{n,k}^\epsilon$ that

$$\mathbf{P}\left[\mathcal{T} \neq \left(\frac{1+5\delta}{2}\right) T_{\text{cov}}(G)\right] = o(1) \text{ as } n \to \infty. \tag{6.8}$$

Let μ be the probability on $\mathcal{X}(G)$ given by first sampling $\mathcal{R} \subseteq G$ according to μ_0, the measure on subsets of V given by running X to time $(1+5\delta)/2 \cdot T_{\mathrm{cov}}(G)$, then sampling $f|\mathcal{R}$ by marking with iid fair coins and $f|V \setminus \mathcal{R} \equiv 0$. Define $\widetilde{\mu}$ similarly except by sampling $\mathcal{R} \subseteq G$ according to $\widetilde{\mu}_0$, the measure given by running X up to time $\mathcal{T}$ rather than $(1+5\delta)/2 \cdot T_{\mathrm{cov}}(G)$. As a consequence of (6.8),

$$\|\mu - \widetilde{\mu}\|_{TV} \leq \mathbf{P}\left[\mathcal{T} \neq \left(\frac{1+5\delta}{2}\right) T_{\mathrm{cov}}(G)\right] = o(1) \text{ as } n \to \infty.$$

Suppose we have two independent random walks X, X' on G, each with stationary initial distribution, and let $\mathcal{T}, \mathcal{T}'$ be stopping times for each as in (6.7). Using the same notation as the previous proof, by the definition of $\mathcal{T}_2'$ we have

$$\begin{aligned} \mathbf{E}[\mathbf{E}[\exp(\zeta|\mathcal{L} \cap \mathcal{L}' \cap E_\ell|)|\mathcal{G}(E_\ell)]] \leq & \mathbf{E} \prod_{x \in E_\ell} \left(1 + e^{\zeta}\left(\prod_{j=1}^{N(x,\mathcal{T}')} (1 - q_j'(x))\right)\right) \\ \leq & \mathbf{E} \prod_{x \in E_\ell} \left(1 + e^{\zeta}\left(\prod_{j=1}^{N(x)} (1 - q_j'(x))\right)\right) \end{aligned} \tag{6.9}$$

where $N(x) = (1+2\delta)\log|H_{n,k}^{\epsilon}|/2\overline{p}_{r,R}(x)$ and k is such that $x \in H_{n,k}^{\epsilon}$. Let

$$\widetilde{N}(x) = (1-\delta)N(x) \geq \frac{(1+\delta/2)\log|H_{n,k}^{\epsilon}|}{2\overline{p}_{r,R}(x)}.$$

Observe that (6.9) is bounded by

$$\mathbf{E} \prod_{x \in E_\ell} \left(1 + e^{\zeta}\left(\prod_{j=1}^{\widetilde{N}(x)} (1 - q_{i(j)}'(x)) + \mathbf{1}_{\{I(\widetilde{N}(x)) > N(x)\}}\right)\right)$$

As E_ℓ satisfies the hypotheses of Lemma 6.2, this is in turn bounded by

$$\begin{aligned} & \mathbf{E} \prod_{x \in E_\ell} \left(1 + e^{\zeta}\left(\prod_{j=1}^{\widetilde{N}(x)} (1 - (1-\delta/4)Q_j'(x))\right) + O(|V|^{-100})\right) \\ \leq & \exp\left(e^{\zeta} \sum_k |H_{n,k}^{\epsilon}|^{-\delta}\right). \end{aligned}$$

The theorem now follows from Hölder's inequality, as in the previous proof. $\square$

6.4. *The Lamplighter.*

Proof of Theorem 1.5. This is proved by making several small modifications to the proof of Theorem 1.3. Namely, rather than considering the range of the random walk run up to time $\mathcal{T}$ as in either (6.4) or (6.7), one considers the range $\mathcal{R}(x)$ run up to time $\mathcal{T}$, then conditioned on hitting a given point x. Exactly the same argument shows that the total variation distance of the law $\widetilde{\mu}_x$ on markings $\mathcal{X}(G)$ induced by iid coin flips on $\widetilde{\mathcal{R}}(x)$ from the uniform measure on $\mathcal{X}(G)$ is $o(1)$. This implies that the law μ_x on markings of $\mathcal{X}(G)$ given by iid coin flips on the range $\mathcal{R}(x)$ run up to time $\frac{1+\epsilon}{2}T_{\rm cov}(G_n)$, and conditioned to hit x, and the uniform measure is $o(1)$. At time $\frac{1+2\epsilon}{2}T_{\rm cov}(G_n)$, the random walk is well mixed conditional on its position at $\frac{1+\epsilon}{2}T_{\rm cov}(G_n)$, from which the result is clear. $\square$

7. Further Questions.

1. Theorem 1.3 yields a wide class of examples where the threshold for indistinguishability is at $\frac{1}{2}T_{\rm cov}$, and $\mathbf{Z}_n^2$ is an example where the threshold is at $T_{\rm cov}$. Does there exists a sequence (G_n) of vertex transitive graphs where the threshold is at $\alpha T_{\rm cov}(G_n)$ for $\alpha \in (1/2, 1)$?
2. Our statistical test for uniformity is only valid for $\alpha > 1/2$. For $\alpha \leq 1/2$, the natural reference measure is iid markings conditioned on the number of zeros being on the order of $|V|^{1-\alpha}$. Can analogous results be proved in this setting?
3. Our definition of uniform local transience is given in terms of the Green's function summed up to the *uniform mixing time*. Does it suffice to assume only the uniform decay of
$$g(x,y;G) = \sum_{t=1}^{T} p^t(x,y;G)$$
where $T = T_{\rm mix}(G)$ or even $T = T_{\rm rel}(G)$?
4. The complete graph K_n does not satisfy the hypotheses of Theorem 1.3 yet the lamplighter walk on K_n has a threshold at $\frac{1}{2}T_{\rm cov}(K_n)$. Is there a more general theorem allowing for a unified treatment of this case?

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